Lecture 11: Confidence Intervals Based on a Single Sample

MSU-STT-351-Sum-19A

Interval Estimation

Suppose $X = (X_1, \ldots, X_n)$ is a random sample from $F(x|\mu)$, where μ is the unknown parameter. A point estimator of μ is a **value** obtained from the sample. For example, if μ is the population mean, then the sample mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is called an estimator of μ. When the sample is from Bernoulli population $B(1, p)$, then sample proportion $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is an estimator of p. Note $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$, obtained from a specific data set $x = (x_1, \ldots, x_n)$ is called an estimate of μ.

A confidence interval for μ is an interval $[L(X), U(X)]$ which has a high probability of containing μ. That is,

$$P(\mu \in [L(X), U(X)]) = 1 - \alpha,$$

where α is usually small ($\alpha = 0.05$ or 0.01).

Confidence Intervals Based on a Single Sample

1. Normal Population with Known σ
Assume we have a random sample $X = (X_1, \ldots, X_n)$ from $N(\mu, \sigma)$, where σ is known and μ is unknown.

(a) Two-sided confidence interval for μ
Let $\alpha = 0.05$. Since X_i's follow normal with unknown μ and known σ,

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

Since $P(Z > 1.96) = P(Z < -1.96) = 0.025 = \frac{\alpha}{2}$, we have

$$P\Big(-1.96 < \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < 1.96\Big) = 0.95$$

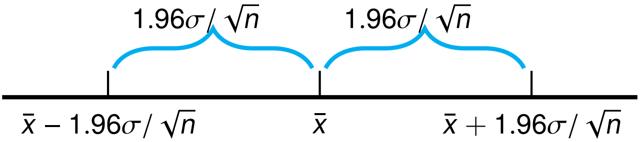
which implies that

$$P\Big(\overline{X} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}\Big) = 0.95$$

Confidence Intervals Based on a Single Sample

Thus, the 95% confidence interval for μ, based on observed $x = (x_1, \cdots, x_n)$, is

$$\left[\bar{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \ \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right].$$



Similarly, $100(1-\alpha)\%$ (two-sided) confidence interval for μ (when σ is known) is

$$\left[\bar{x} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}},\ \ \bar{x} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right],$$

where $P(Z > z_{\alpha/2}) = P(Z < -z_{\alpha/2}) = \frac{\alpha}{2}$.

(b) One-sided intervals

Sometimes, we are only interested in obtaining an **upper** or a **lower** bound for the parameter. Then the only difference is that we use z_α instead of $z_{\alpha/2}$ as the multiplier. Thus one-sided intervals are:

$$\mu < \overline{x} + z_\alpha \frac{\sigma}{\sqrt{n}} = U(x), \text{ or } \mu > \overline{x} - z_\alpha \frac{\sigma}{\sqrt{n}} = L(x),$$

where $P(Z \geq z_\alpha) = \alpha$.

3. Other levels of confidence.
As seen earlier, the multiplier 1.960 is found from probabilities of the standard normal distribution. Also, the $100(1-\alpha)\%$ confidence interval on μ (when σ is known), based on the data $x_1, \cdots, x_n$, is

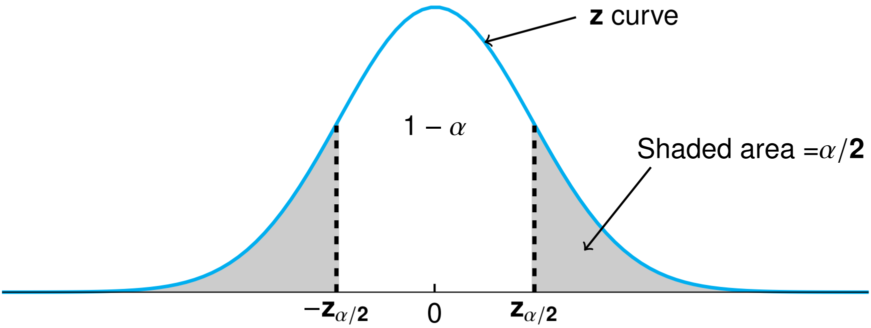
$$\overline{x} \pm z_{\alpha/2}\sigma/\sqrt{n}.$$

Confidence Intervals Based on a Single Sample

Some common values of z_α such that $P(Z \geq z_\alpha) = \alpha$ are:

$100(1-\alpha)\%$	α	$\alpha/2$	$z_{\alpha/2}$
80%	0.200	0.100	1.282
90%	0.100	0.050	1.645
95%	0.050	0.025	1.960
99%	0.010	0.005	2.576

That is, if $Z \sim N(0,1)$, then $P(Z > z_\alpha) = \alpha$. The critical value z_α can be obtained from Table $A.3$.



Example 2 (Ex 6): The yield point of a particular type of mild steel-reinforcing bar is known to be **normally** distributed with $\sigma = 100$. The composition of the bar has been slightly modified, but the modification is not believed to have affected either the normality or the value of σ.
(a) Assuming this to be the case, if a sample of 25 modified bars resulted in a sample average yield point of 8439 lb, compute a 90% CI for the true average yield point of the modified bar.
(b) How would you modify the interval in part (a) to obtain a confidence level of 92%?
Solution: (a): $\sigma = 100$; $(1 - \alpha) = 0.90 \Rightarrow \frac{\alpha}{2} = 0.05$ and $z_{\alpha/2} = 1.645$. Hence, the CI for μ is

$$8439 \pm \frac{(1.645)(100)}{\sqrt{25}} = 8439 \pm 32.9 = (8406.1, 8471.9).$$

(b) For $1 - \alpha = 0.92 \Rightarrow \alpha = 0.08 \Rightarrow \alpha/2 = 0.04$, $z_{\alpha/2} = z_{0.04} = 1.75$

Confidence Interval for Exponential Mean

Example 1. Let $X_1, \ldots, X_n$ be *iid* exponential $Exp(\lambda) = G(1, 1/\lambda)$. Then

$$\sum_1^n X_i \sim G(n, 1/\lambda) \Leftrightarrow 2\lambda \sum_1^n X_i \sim G(n, 2) \equiv \chi^2_{2n} = \chi^2_\nu,$$

the χ^2 distribution with ν degrees of freedom ($d.f.$).

Consider $\alpha = 0.05$. Let A and B (both positive) be such that

$$P\Big(A < 2\lambda \sum_1^n X_i < B\Big) = .95$$

$$\Rightarrow P\Big(\frac{2\sum_1^n X_i}{B} < \frac{1}{\lambda} < \frac{2\sum_1^n X_i}{A}\Big) = .95.$$

Thus $\left(\frac{2\sum_1^n X_i}{B}, \frac{2\sum_1^n X_i}{A}\right)$ is a 95% CI for the mean $E(X) = \frac{1}{\lambda}$ of exponential distribution.

The values of A and B can be obtained from Table $A.7$ as
$P(\chi^2_\nu > B) = 0.025$ ($= \frac{\alpha}{2}$ in general) and
$P(\chi^2_\nu > A) = 0.975$ ($= 1 - \frac{\alpha}{2}$ in general).

Let $\chi^2_{\nu,\alpha}$ be the critical value for the upper tail area of χ^2_ν distribution so that

$$P(\chi^2 > \chi^2_{\nu,\alpha}) = \alpha.$$

This implies

$$P(\chi^2_{\nu,1-\alpha/2} < 2\lambda \sum_{i=1}^{n} X_i < \chi^2_{\nu,\alpha/2}) = 1 - \alpha,$$

which gives $100(1-\alpha)\%$ CI for $\frac{1}{\lambda}$ as

$$\Big(\frac{2\sum_{i=1}^{n} X_i}{\chi^2_{\nu,\alpha/2}}, \ \ \frac{2\sum_{i=1}^{n} X_i}{\chi^2_{\nu,1-\alpha/2}}\Big).$$

Example 3 (Ex 10): A random sample of $n = 15$ heat pumps of a certain type yielded the following observations on lifetime (in years):

2.0, 1.3, 6.0, 1.9, 5.1, 0.4, 1.0, 5.3,
15.7, 0.7, 4.8, 0.9, 12.2, 5.3, 0.6.

(a) Assuming lifetimes follow exponential $Exp(\lambda)$ distribution, obtain 95% CI for $E(X) = 1/\lambda$.

(b) How it should be altered to obtain 99% CI?

(c) What is 95% CI for SD(X)?

Solution: (a) When $n = 15$, $2\lambda \sum_{i=1}^{n} X_i$ has a chi-squared distribution with 30 *df*. From the 30 *df* row of Table A.7, the critical values that capture lower and upper tail areas of 0.025 (and thus a central area of .95) are 16.791 and 46.979, respectively. Using the formula in Example 1, the 95% CI for $\mu = \frac{1}{\lambda}$ is

$$\left(\frac{2\sum_{i=1}^{n} X_i}{46.979}, \frac{2\sum_{i=1}^{n} X_i}{16.791}\right).$$

Since $\sum_{i=1}^{n} X_i = 63.2$, the interval is $(2.69, 7.53)$.

(b) For 99% confidence level, the critical values that capture area .005 in each tail of the chi-squared curve with 30 d.f. are 13.787 and 53.672.

(c) Note $Var(X) = \frac{1}{\lambda^2}$, when $X \sim Exp(\lambda)$ distribution. So, the standard deviation is $\frac{1}{\lambda}$ which is same as the mean. Thus, the interval of (a) is also a 95% C.I. for the standard deviation of the lifetime distribution.

Level, Precision and Sample Size

Level: The length of the confidence interval increases with the multiplier $z_{\alpha/2}$ (see normal CI's). The only 100% confidence interval is $(-\infty, \infty)$.

Sample size: Suppose we know σ and want to find the sample size n required to obtain a **specified width** for a confidence interval. For example, if $\sigma = 25$, what is the n required to have a 95% CI having width at most 10?

Note the width

$$10 = 2(1.960(25/\sqrt{n})) \Rightarrow n = [2(1.96)(25/10)]^2 = 96.$$

The general expression for the n to give a width w in a $100(1-\alpha)\%$ CI is

$$n = \left(2z_{\alpha/2}\frac{\sigma}{w}\right)^2.$$

Confidence Intervals Based on a Single Sample

Large Sample CI's for Population Mean μ, When σ is unknown.

For large n, we may assume/expect that $S \approx \sigma$. In other words, $Z = \frac{\overline{X}-\mu}{S/\sqrt{n}}$ has approximately a standard normal distribution, giving the interval $\overline{x} \pm z_{\alpha/2} S/\sqrt{n}$.

(i) Two-sided CI for μ when n is large

Let $X = (X_1, \ldots, X_n)$ be a random sample from $F(x|\theta)$. Let $P[L(X) \leq \theta \leq U(X)] = 1 - \alpha$.

Then as seen earlier $I = [L(x), U(x)]$ is called $100(1-\alpha)\%$ confidence interval for θ.

Example 4. Suppose $X = (X_1, \ldots, X_n)$ is a random sample from a population with mean μ and variance σ^2 (both parameters are unknown). Find $100(1-\alpha)\%$ CI for μ.

Solution:

Let $\overline{X} = \frac{1}{n}\sum_1^n X_i$ and $S^2 = \frac{1}{n-1}\sum_1^n (X_i - \overline{X})^2$ be sample mean and sample variance. For large n (and using CLT), $\overline{X} \sim N\Big(\mu, \frac{\sigma}{\sqrt{n}}\Big)$. If σ is not known, it can be estimated and replaced by S.

Hence, for large n,

$$\overline{X} \sim N\Big(\mu, \frac{S}{\sqrt{n}}\Big) \Leftrightarrow \sqrt{n}(\overline{X} - \mu)/S \sim N(0, 1).$$

Aim is to find $L(x)$ and $U(x)$ such that

$$P(L(X) < \mu < U(X)) = 1 - \alpha.$$

We know

$$P\left(-z_{\alpha/2} < \sqrt{n}\frac{\overline{X}-\mu}{S} < z_{\alpha/2}\right) = (1-\alpha).$$

Solving for μ we have

$$P\left(\overline{X} - z_{\alpha/2}\frac{S}{\sqrt{n}} \leq \mu \leq \overline{X} + z_{\alpha/2}\frac{S}{\sqrt{n}}\right) = 1-\alpha.$$

Thus, $100(1-\alpha)\%$ CI for μ, when σ^2 **unknown**

$$\left[\overline{X} - z_{\alpha/2}\frac{S}{\sqrt{n}},\ \ \overline{X} + z_{\alpha/2}\frac{S}{\sqrt{n}}\right].$$

Example 5. Find a 95% CI for μ, when σ is unknown. Note

$$100(1-\alpha) = 95 \Rightarrow (1-\alpha) = .95 \Rightarrow \frac{\alpha}{2} = 0.025 \Rightarrow z_{0.025} = 1.96.$$

Hence, a 95% CI for μ is

$$\left[\overline{X} - 1.96\frac{S}{\sqrt{n}},\ \ \overline{X} + 1.96\frac{S}{\sqrt{n}}\right].$$

The following table values for $N(0,1)$ distribution are useful:

$100(1-\alpha)\%$	$z_{\alpha/2}$
90%	1.645
95%	1.96
99%	2.575

Definition: The half-width $B = z_{\alpha/2} \frac{S}{\sqrt{n}}$ is called **the bound on the error** of estimation. For 95% CI, we have $B = 1.96 \frac{S}{\sqrt{n}}$.

How to choose the sample size?

Suppose we want n such that the amount of error, with 95% confidence, is B (specified). In that case,

$$B = 1.96 \frac{S}{\sqrt{n}} \Rightarrow n = \left[\frac{1.96 s}{B} \right]^2$$

Example 6 (Ex 14):

The following is the summary information for the fracture strengths (MPa) of $n = 169$ ceramic bars fired in a particular kiln: $\overline{x} = 89.10,\ s = 3.73$.

(a) Calculate a (two-sided) confidence interval for average fracture strength using a confidence level of 95%. Does it appear that true average fracture strength has been precisely estimated?

(b) Suppose the investigators had believed a priori that the population standard deviation was about a 4 MPa. Based on this supposition, how large a sample would have been required to estimate μ to within .5 MPa with 95% confidence?

Solution: (a) The two-sided CI for μ is

$$89.10 \pm 1.96 \frac{3.73}{\sqrt{169}} = 89.10 \pm .56 = (88.54, 89.66).$$

Yes, this is a very narrow interval. It appears quite precise.

(b) The required sample size is

$$n = \left[\frac{(1.96)(4)}{.5}\right]^2 = 245.86 \Rightarrow n = 246.$$

(ii) One-sided CIs for μ, when n is large and σ unknown.

We want L and U such that

$$P[\mu > L(X)] = 1 - \alpha, \text{ or } P[\mu < U(X)] = 1 - \alpha,$$

for a specified α.
Now

$$P\Big[\frac{(\overline{X} - \mu)\sqrt{n}}{S} \leq z_\alpha\Big] = 1 - \alpha$$

$$\Rightarrow P\Big[\mu \geq \overline{X} - z_\alpha \frac{S}{\sqrt{n}}\Big] = 1 - \alpha.$$

Thus, the large sample lower $100(1 - \alpha)\%$ confidence bound is

$$\mu \geq \overline{X} - z_\alpha \frac{S}{\sqrt{n}}.$$

Confidence Intervals Based on a Single Sample

The following table for z_α is useful.

$100(1-\alpha)\%$	z_α
90%	1.28
95%	1.645
99%	2.33

The 95% upper confidence bound for μ is

$$\mu \le \overline{X} + z_\alpha \frac{S}{\sqrt{n}} = \overline{X} + 1.645 \frac{S}{\sqrt{n}}.$$

Example 6: The charge-to-tap time (min) for a carbon steel in one type of open hearth furnace was determined for each heat in a sample of size 36, resulting in a sample mean time of 382.1 and a sample standard deviation of 31.5. Calculate a 95% upper confidence bound for true average charge-to-tap time.

Solution:
A 95% upper confidence bound for the true average charge-to-tap time is:

$$\overline{x} + (1.645)\left(\frac{S}{\sqrt{n}}\right) = 382.1 + (1.645)\left(\frac{31.5}{\sqrt{36}}\right) = (382.1 + 8.64) = 390.74$$

That is, with 95% confidence, the value of μ lies in the interval (0 min, 390.74 min).

Confidence Intervals for the Population Proportion p

Let $X = 1$ represent a characteristic of the population units and $p = P(X = 1)$ denote the population proportion having that characteristic. Based on the data $X_1, \ldots, X_n$ on X, let S_n denote the number of units in the sample possessing that characteristic property. Note then S_n follows binomial with mean np and Variance $np(1-p)$. That is,

$$E(S_n) = p(1-p) : \quad Var(S_n) = np(1-p).$$

Also, an estimate of p is the sample proportion, namely, $\hat{p} = S_n/n$. Then

$$E(\hat{p}) = p : \quad var(\hat{p}) = p(1-p)/n; \;\; \sigma_{\hat{p}} = \sqrt{p(1-p)/n}.$$

Hence, when n is large,

$$\hat{p} \approx N\left(p, \; \sqrt{p(1-p)/n} \;\right).$$

This fact can be used to obtain confidence interval for p.

Confidence Intervals for the Population Proportion p

Note that

$$P\left(-z_{\alpha/2} \leq \frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \leq z_{\alpha/2}\right) \approx (1-\alpha).$$

Solving now the quadratic equation in p, namely,

$$\hat{p}-p = z_{\alpha/2}\sqrt{p(1-p)/n}$$

gives us two roots given by

$$p = \frac{\hat{p} + \frac{z^2_{\alpha/2}}{2n} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n} + \frac{z^2_{\alpha/2}}{4n^2}}}{1+(z^2_{\alpha/2})/n}$$

which forms the end-points of confidence interval for p.

Large sample interval. The **large sample** confidence interval for p is obtained by ignoring the terms $z^2/2n$ and $z^2/4n^2$, which leads to

$$\hat{p} \pm z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n}.$$

Example 7: In inspecting the quality of soil compaction in a highway project, 10 out of 50 specimens do not pass the CBR requirement. Estimate the actual proportion p of embankment that will be compacted (that is, meets the CBR requirement) and also find the 95% confidence interval for p. (10 points)

Solution:

The point estimate for p is given by $\hat{p} = \frac{40}{50} = 0.8$. The 95% confidence interval for p is

$$\left[0.8 - 1.96\sqrt{\frac{0.8(1-0.8)}{50}},\ 0.8 + 1.96\sqrt{\frac{0.8(1-0.8)}{50}}\right] = [0.69,\ 0.91].$$

Home work:

Sec 7.1: 3, 9, 11

Sec 7.2: 15, 22, 26