Equilibrium Determinacy With Behavioral Expectations

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Abstract

Behavioral expectations affect determinacy in macroeconomic models. Relaxing rational expectations can make models more or less well behaved, depending on the behavioral assumptions. In some cases, multiplicity is created; in other cases, multiplicity is eliminated. Is it possible to tell exactly when there are multiple solutions? Yes: I derive a Behavioral Blanchard-Kahn sufficient condition that ensures a unique equilibrium exists. If and only if this condition or a Sunspot Admissibility condition hold, then a model's solution must be unique. These conditions depend on the spectrum of the behavioral expectation operator. I describe how to check these conditions for an arbitrary behavioral expectation, and illustrate with a large variety of popular types of expectations, heuristics, and information frictions. As an example, I demonstrate that a large class of behavioral expectations imply a unique solution to the New Keynesian model with an interest rate peg, including all strictly backwards-looking heuristics. Another class of expectations imply that asset prices exhibit non-fundamental volatility in a standard model.

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1 Introduction

When do dynamic models have unique solutions? Answering this question is crucial in macroeconomics. Models with unique solutions make clear, robust predictions. When models have multiple solutions, inference is more challenging, although these models can still be useful for understanding dynamics in the presence of extrinsic volatility. The question has practical importance. For example, the main motivation for modern monetary policy's "Taylor Principle" is to eliminate multiplicity. Fortunately, whether models have unique solutions is known exactly when agents have rational expectations: the Blanchard-Kahn condition (Blanchard and Kahn, 1980) is necessary and sufficient for a unique solution to exist.

However, when expectations are nonrational, the answer changes. The Blanchard-Kahn condition only applies to full information rational expectations (FIRE) models. Under some types of behavioral expectations, models that have unique solutions under FIRE will have multiple solutions. For other types of behavioral expectations, models that would have multiplicity under FIRE will have a unique solution. Behavioral expectations can fundamentally change equilibrium properties. The traditional Blanchard-Kahn condition cannot necessarily be applied, even if the behavioral model can be rewritten as a FIRE model.

To resolve these issues, I introduce a *Behavioral Blanchard-Kahn* condition. Like the original, it is a condition on a model's generalized eigenvalues that ensures a unique solution exists: the number of non-predetermined variables must equal the number of unstable eigenvalues. The behavioral modification is that "unstable eigenvalues" are now defined as those that are larger in magnitude than the *spectral radius of the expectation operator*. For rational expectations, the spectral radius is one, implying the classic Blanchard-Kahn condition. But different types of behavioral expectations feature other values.

After proving that the main Behavioral Blanchard-Kahn condition is sufficient for a unique solution to exist, I explore when it is necessary. It almost always is under rational expectations, but this is not true more generally. Instead, I prove that *solutions are unique if and only if the Behavioral Blanchard-Kahn or a Sunspot Admissibility condition hold.* When a model is underdetermined, rational expectations models feature a multiplicity of sunspot equilibria. But crucially, many behavioral expectations do not admit sunspot equilibria, while others only admit them for a subset of models. This is because sunspot equilibria are driven by self-fulfilling forecast errors. But with behavioral expectations, arbitrary forecast errors can be challenging or even impossible to construct.

These results are valuable for understanding how behavioral expectations affect the theoretical properties of equilibria, but they are also useful for practitioners. Many behavioral models can be rewritten in terms of a FIRE model; typically, practitioners will do so, and then check the traditional Blanchard-Kahn condition with a computer. Changing the type of behavioral expectation – or even just adjusting its parameter values – can eliminate or attain multiplicity. How can one know *ex ante* whether changing assumptions or parameterization will affect equilibrium uniqueness? The Behavioral Blanchard-Kahn condition provides the answer.

The theoretical results also have practical applications. For example, I demonstrate that a simple asset pricing model with standard parameters can feature multiple equilibria and extrinsic volatility for a class of behavioral expectations. In this setting, government policy may be needed to stabilize the market. Conversely, in other settings government policy designed to resolve a rational expectations multiplicity may be unnecessary. Gabaix (2020) argues that this is the case in the New Keynesian model, for a sufficiently large behavioral bias. I generalize this result, demonstrating that equilibria are unique under an interest rate peg for a variety of behavioral expectations, including *all* backwards-looking heuristics.

The main results in this paper apply to models where agents are *sophisticated*: when forecasting, they form expectations over the equilibrium time series. However, there is not consensus in the literature that this an appropriate assumption. Alternatively, agents might be *naive*: they forecast as if rational expectations will hold in the future. I adapt the main theorem for these types of models too; if the traditional Blanchard-Kahn condition holds for the rational version of a model, then the corresponding model where naive agents have behavioral expectations is guaranteed to have a unique solution.

After exploring how determinacy depends on the spectral radius and eigenvalues of expectations operators in general, I characterize these features for a variety of behavioral expectations appearing in the literature. Figure 1 reports how equilibrium uniqueness relates to the traditional Blanchard-Kahn condition for a subset of these expectations. The uniqueness properties vary dramatically, but this is not a case where "anything goes": each type of behavioral expectations implies falsifiable

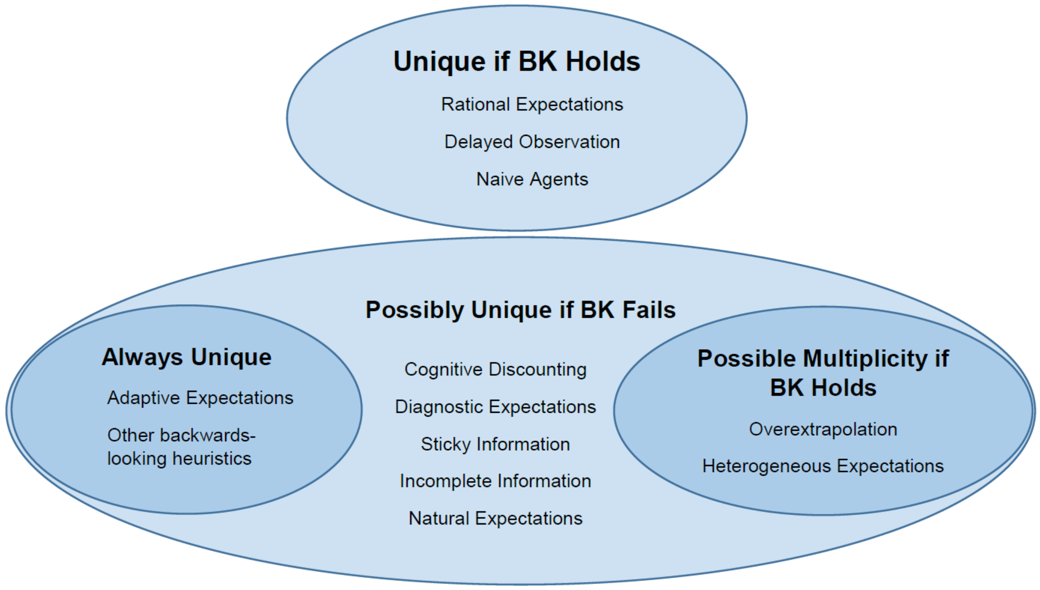


Figure 1: Multiplicity and the Traditional Blanchard-Kahn Condition

Notes: The diagram classifies a non-exhaustive variety of behavioral expectations based on equilibrium uniqueness when the traditional Blanchard-Kahn (BK) condition holds or fails. Some types of expectations for which equilibria are *possibly unique* when the condition fails may also feature *possible multiplicity* even if the condition holds. For a smaller subset of expectations, equilibria are *always unique* if they exist. Each type is defined and classified in Section 7, except for "Naive Agents" who mistakenly believe that rational expectations will hold in the future (Section 6).

predictions, and has exact conditions for when equilibria are unique. In all cases, I derive their spectral properties analytically, but also describe how the spectral radius can be calculated numerically if an analytical solution is impossible. Furthermore, I show how to represent a simple subset of incomplete information models and dynamic beauty contests as behavioral expectations, and describe their spectra too.

The existence and uniqueness theorems apply to a large class of behavioral expectations. But there are some restrictions. First, they must be linear, which applies to most behavioral expectations in the literature, but not all. Second, I focus on stationary models, which excludes behavioral learning processes, as in Evans and Honkapohja (2012). Third, the expectations must be "series-agnostic," so that agents apply the same expectation operator to all time series in the model. Some evidence suggests that forecasts of different macroeconomic series feature different biases (Bordalo, Gennaioli, Ma, and Shleifer, 2020), which may not fit in my framework. Moreover, my representation can only account for limited heterogeneity; if agents are not symmetric, then the model must be able to be written in terms of average forecasts or some other single expectation. This works for some models with informational heterogeneity (Woodford, 2003) or behavioral heterogeneity (Branch and McGough, 2004), but will not apply to heterogeneous expectations in general.

This paper joins a large literature that derives existence and uniqueness conditions for specific models without rational expectations. Some are special cases of this paper's general results, while others have features that do not fit in my framework.A growing literature explores how specific types of behavioral expectations affect determinacy in the New Keynesian model. Some types of behavioral expectations expand the region of monetary policy parameters that ensure determinacy; this includes the cognitive discounting in Gabaix (2020), the heuristic switching in Bertasiute, Massaro, and Weber (2020), and the dispersed information structure in Gallegos (2022). In other cases, behavioral expectations reduce the determinacy region, so that a standard Taylor rule is not reactive enough; this is the case for heterogeneous expectations models studied in Branch and McGough (2009), Massaro (2013), and Anufriev, Assenza, Hommes, and Massaro (2013). Other expectation-formation frictions can eliminate multiplicity entirely, as with the level-k thinking studied by García-Schmidt and Woodford (2019) and Farhi and Werning (2019), or the imperfect memory in Angeletos and Lian (2021).

The remainder of the paper is organized as follows. Section 2 describes how to represent behavioral expectations as infinite-dimensional linear operators. Section 3 presents the main theorems. Section 4 demonstrates why the Behavioral Blanchard-Kahn condition delivers uniqueness in a simple asset pricing model. Section 5 examines determinacy in the behavioral New Keynesian model. Section 6 considers existence and uniqueness when agents are naive rather than sophisticated. Section 7 characterizes the spectra of many types of behavioral expectations.

2 Expectations as Operators and Other Notation

In this section I describe how to define a general class of behavioral expectations as infinite-dimensional operators. I discuss the spectral radius, a characteristic of operators that determines whether dynamic models with behavioral expectations have unique solutions.

2.1 Information Bases and Behavioral Expectations

Consider an arbitrary demeaned stationary time series X_t. Assume that the time series can be represented as a moving average in terms of some underlying mean-zero white noise process ω_t with square-summable coefficients X_j:

$$x_t = \sum_{j=0}^{\infty} X_j \omega_{t-j} \tag{1}$$

$$= X(L)\omega_t$$

where $X(L)$ is the corresponding lag-operator polynomial. The coefficients X_j are $m \times n$ matrices, where m is the dimension of the time series X_t, and n is the dimension of the white noise process ω_t.

The white noise ω_t represents stochastic shocks. Crucially, when considering expectations, ω_t is assumed to be in a forecaster's information set at time t and beyond. This assumption sidesteps concerns about the invertibility of $X(L)$ and fundamentalness of the time series x_t.

Different forms of expectations are indexed by b. The type-b expectation of a time series h periods into the future is written as $\mathbb{E}_t^b[X_{t+h}]$; when written without a b, $\mathbb{E}_t$ indicates the rational expectation. The subscript t denotes that the expectation is conditional on the period t information set (i.e. all shocks ω_s for $s \leq t$.) Regardless of the type of behavioral expectations, I assume that agents perfectly forecast variables in their information set:

$$\mathbb{E}_t^b[X_t] = X_t$$

Thus if an agent builds a factory today, they will expect to have that factory tomorrow. This assumption is so that laws of motion hold with equality even when expectations are applied. But it introduces an additional challenge: behavioral expectations become nonlinear.

2.2 Time Series as Vectors and Expectations as Operators

A stationary time series of form (1) can be represented as an $\infty \times n$ block vector $\vec{X}$, where n is the dimension of the white noise ω_t

$$\vec{X} \equiv \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \end{pmatrix}$$

I consider behavioral expectations that are *linear in one-period-ahead time series.*[1] When this is true, an arbitrary behavioral expectations operator can be represented as an infinite dimensional matrix – technically, a bounded linear operator – which operates on infinite dimensional vector spaces (ℓ^2 in this case).

For b-type expectations with a one period horizon, i.e. $\mathbb{E}_t^b[X_{t+1}]$, the corresponding operator $\mathcal{E}_b$ is defined by its block matrix representation

$$\mathcal{E}_b = \begin{pmatrix} \mathcal{E}_{b,0,0} & \mathcal{E}_{b,0,1} & \mathcal{E}_{b,0,2} & \cdots \\ \mathcal{E}_{b,1,0} & \mathcal{E}_{b,1,1} & \mathcal{E}_{b,1,2} & \ddots \\ \mathcal{E}_{b,2,0} & \mathcal{E}_{b,2,1} & \mathcal{E}_{b,2,2} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{2}$$

for conformability, the blocks $\mathcal{E}_{b,i,j}$ must be $m \times m$. When written without a b, $\mathcal{E}$ denotes the rational expectation operator.

I assume throughout that expectations are *series-agnostic*, which means that however the expectations operate on a single time series is applied to all time series symmetrically.[2] For example, if agents forecast GDP with a form of behavioral expectations, they must also forecast inflation with the same form of behavioral expectations. Mathematically, this assumes that the expectations operator commutes with

[1]Section 7 demonstrates that many popular types of expectations fulfill this property. I also point out some examples that do not, and consider linear analogs.

[2]This is equivalent to assumption (A3) in Branch and McGough (2018), and Lemma 1 in Gabaix (2020). Once consequence of the series-agnostic property is that this model structure can only accommodate certain forms of heterogeneity; the macroeconomic model must be able to be written in terms of a single expectation operator. In some cases $\mathbb{E}_t^b$ can represent average expectations, such as in Mankiw and Reis (2002) or Woodford (2003) where agents have heterogeneous information, or in Branch and McGough (2009) where different types of agents form expectations in different ways.

arbitrary matrices, which implies that the $\mathcal{E}_{b,i,j}$ blocks are proportional to the $m \times m$ identity:

Property 1 *Series-agnostic expectations of type b satisfy:*

1. *For any conformable matrix B,*

$$\mathbb{E}_t^b[BX_{t+h}] = B\mathbb{E}_t^b[X_{t+h}]$$

2. *The general operator representation $\mathcal{E}_b$ has blocks*

$$\mathcal{E}_{b,i,j} = \alpha_{b,i,j} I$$

for some scalar $\alpha_{b,i,j}$

The series-agnostic property is useful for proving the main theorems. Without the property, it is still possible to represent expectations as a general operator as in equation (2), but characterizing general properties of equilibrium becomes challenging without imposing additional structure on the expectations or the economic model.

In order to simplify notation, when finite matrices appear in operator equations, they represent the infinite operator with the finite matrix repeating on the main block diagonal. Thus for any conformable matrix B:

$$B\vec{X} = \begin{pmatrix} BX_1 \\ BX_2 \\ BX_3 \\ \vdots \end{pmatrix} \qquad B\mathcal{E}_b = \begin{pmatrix} B\mathcal{E}_{b,0,0} & B\mathcal{E}_{b,0,2} & B\mathcal{E}_{b,0,2} & \dots \\ B\mathcal{E}_{b,1,0} & B\mathcal{E}_{b,1,2} & B\mathcal{E}_{b,1,2} & \ddots \\ B\mathcal{E}_{b,2,0} & B\mathcal{E}_{b,2,2} & B\mathcal{E}_{b,2,2} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Similarly, when scalars appear in operator equations, it is implied that they are multiplied by the identity, e.g. $2 + \mathcal{E}$ is equivalent to $2I + \mathcal{E}$.

The lag operator L has a block Toeplitz representation:

$$L = \begin{pmatrix} 0 & 0 & 0 & \dots \\ I & 0 & 0 & \ddots \\ 0 & I & 0 & \ddots \\ 0 & 0 & I & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{3}$$

The lag operator shifts vectors by one block. It is an isometry, left-invertible by its transpose, but not right-invertible:

$$L'L = I \neq LL'$$

2.3 The Spectral Radius

Why is it crucial to consider behavioral expectations as infinite-dimensional operators? Because one well-understood characteristic of an operator is its *spectral radius*, and in many cases the necessary and sufficient condition for a model to have a unique solution depends on the spectral radius of the expectations operator.

The spectral radius of a bounded linear operator is the supremum of the absolute values of its spectrum:

Definition 1 *The* ***spectral radius*** *of an operator* $r(\mathcal{E}_b)$ *is*

$$r(\mathcal{E}_b) \equiv \sup |\lambda|$$

$$s.t. \ \ (\lambda - \mathcal{E}_b)^{-1} \ \ \textit{does not exist}$$

I denote the spectral radius of an expectation operator by $r(\mathcal{E}_b)$. In this paper, the most useful property of the spectral radius is:

Property 2 *For a matrix* B *with largest magnitude eigenvalue* $|\lambda_B| < r(\mathcal{E}_b)^{-1}$ *and series-agnostic expectation operator* $\mathcal{E}_b$*, the operator* $I - B\mathcal{E}_b$ *is invertible.*

Proof: Appendix B.1

The spectral radius is straightforward to derive analytically for many forms of expectations, which I demonstrate in Section 7. And even in cases where an analytical expression is impossible, it is simple to calculate numerically (Section 7.2).

In many cases, an operator's spectral radius is the magnitude of the largest eigenvalue, but sometimes operators have no eigenvalues.[3] If it exists, an eigenvalue λ_b of an operator $\mathcal{E}_b$ has associated eigenvector $\vec{y}$ such that $\lambda_b \vec{y} = \mathcal{E}_b \vec{y}$. The eigenvector is the vector representation of the associated *eigenseries*:

Definition 2 *The time series y_t is an* ***eigenseries*** *of type b expectations if there exists an eigenvalue λ_b such that*

$$\lambda_b y_t = \mathbb{E}_t^b[y_t]$$

The eigenseries of behavioral expectations are crucial for determining whether models feature sunspot equilibria.

2.4 Recursive Expectations and the Spectral Radius

The spectral radius is used in the main existence and uniqueness theorems to determine when a recursive expectational equation has a unique fixed point. Before moving on to the general uniqueness theorems, it is useful to prove an intermediate step that depends on the spectral radius.

Let $\mathbb{E}_{t,t+j}^b[x_{t+j+1}]$ denote the iterated expectation

$$\mathbb{E}_{t,t+j}^b[x_{t+j+1}] \equiv \mathbb{E}_t^b \mathbb{E}_{t+1}^b \mathbb{E}_{t+2}^b ... \mathbb{E}_{t+j}^b \left[x_{t+j+1}\right]$$

$\mathbb{E}_{t,t+j}^b[x_{t+j+1}]$ appears in the following Lemma, because the law of iterated expectations does not always apply to behavioral expectations.[4]

Lemma 1 *If the matrix B has largest magnitude eigenvalue $|\lambda_B| < r(\mathcal{E}_b)^{-1}$, the time series y_t is stationary, and the time series x_t satisfies the recursive equation*

$$x_t = \mathbb{E}_t^b \left[y_{t+1}\right] + B\mathbb{E}_t^b \left[x_{t+1}\right]$$

[3] This is the case for the lag operator, as well as for many of the heuristic expectations considered in Section 7.

[4] In Section 6, I consider a "naive" alternative where agents future exogenous variables instead of endogenous forecasts; in such a model, iterated behavioral expectations do not appear.

then x_t is stationary and given by

$$x_t = \sum_{j=0}^{\infty} B^j \mathbb{E}^b_{t,t+j}[y_{t+j+1}]$$

Proof: Appendix B.2

3 Uniqueness in a General Macroeconomic Model

In this section I define a general linear macroeconomic model with behavioral expectations. I define the main Behavioral Blanchard-Kahn condition and prove that when it is satisfied, a unique solutions exists. Then I characterize when the condition is necessary.

3.1 Notation and Definitions

Consider a general linear dynamic stochastic macroeconomic model of the following form. The equilibrium conditions of the model are represented as a single matrix equation:

$$\mathbb{E}^b_t \left[B_{X1} X_{t+1}\right] = B_{X0} X_t + B_Y Y_t \tag{4}$$

$X_t = \begin{pmatrix} K_{t-1} \\ C_t \end{pmatrix}$ is a $n \times 1$ vector of endogenous variables. n_K of the variables are predetermined state variables K_{t-1}, while $n_C = n - n_K$ are control variables C_t. Y_t is a vector of exogenous mean-zero stochastic processes that are realized at time t; $Y_t = Y(L)\omega_t$ is a moving average in the exogenous white noise ω_t. The matrices B_{X0}, B_{X1}, and B_Y encode the equilibrium conditions of the model.[5] The state variable component of X_{t+1} is known exactly at time t, so the expected vector $\mathbb{E}^b_t[X_{t+1}]$ represents

$$\mathbb{E}^b_t[X_{t+1}] = \begin{pmatrix} K_t \\ \mathbb{E}^b_t\left[C_{t+1}\right] \end{pmatrix}$$

[5] This is a general form for most linear macroeconomic models, but some caution is required when adopting behavioral expectations. The modeler cannot be as cavalier about how $\mathbb{E}^b_t\left[B_{X1}X_{t+1}\right]$ is constructed as they can with rational expectations. This is because behavioral expectations are only piecewise linear, and many types do not obey the law of iterated expectations. Theory must inform *which* variables agents forecast. For example, are asset prices determined by forecasts of tomorrow's asset prices? Or by forecasts of the stream of all future dividends? When the law of iterated expectations fail, the answers are not the same.

for any form of expectations, which are always conditional on the information set $\{\omega_{t-j}\}_{j=0}^{\infty}$.[6]

The generalized Schur decomposition of the coefficient matrices is denoted by

$$B_{X0} = QT_0Z \qquad\qquad B_{X1} = QT_1Z$$

where Q and Z are unitary, T_0 and T_1 are upper triangular, and the diagonal of T_0 is arranged so that the generalized eigenvalues are in increasing order. Let ϕ_i denote the ith generalized eigenvalue of the model, i.e. the ratio of diagonal elements $T_{0,i,i}/T_{1,i,i}$. If $T_{1,i,i}$ is zero while $T_{0,i,i}$ is nonzero, the generalized eigenvalue is said to be infinite. If both are zero, then the generalized eigenvalue is said to be undefined.

The generalized eigenvalues with magnitude $|\phi_i| < 1$ are labeled the "stable" eigenvalues. Let n_S denote the number of stable eigenvalues in a model. The generalized eigenvalues with magnitude $|\phi_i| > r(\mathcal{E}_b)$ are labeled "unstable". n_U denotes the number of "unstable" eigenvalues in a model.

Partition the matrices into blocks, separating the first $n - n_U$ dimensions from the remaining n_U dimensions.[7] Denote the partitions as:

$$T_0 = \begin{pmatrix} T_{0,S} & T_{0,SU} \\ 0 & T_{0,U} \end{pmatrix} \qquad T_1 = \begin{pmatrix} T_{1,S} & T_{1,SU} \\ 0 & T_{1,U} \end{pmatrix} \qquad Z = \begin{pmatrix} z_{SK} & z_{SC} \\ z_{UK} & z_{UC} \end{pmatrix}$$

The columns of Z are partitioned into the first n_K dimensions and the remaining n_C dimensions.

I make four regularity assumptions about the model, following Klein (2000) and Blanchard and Kahn (1980).

1. z_{UC} is full rank.

2. B_{X0} and B_{X1} have no undefined generalized eigenvalues.

3. B_{X0} and B_{X1} have no generalized eigenvalues in the interval $[r(\mathcal{E}_b), 1]$.

[6]The assumption that ω_t includes the fundamental shocks driving the exogenous state vector Y_t rules out some sources of potential sunspot equilibria, including information frictions with endogenous signals (Angeletos and Werning, 2006) or model misspecification with hidden state variables (Branch, McGough, and Zhu, 2022). Still, Appendix D.4 demonstrates that some simple information frictions such as forecasting from signals with exogenous noise, are isomorphic to other behavioral expectations that do fit in this framework.

[7]It must be that $n - n_U \geq n_S$, but no assumption is made to rule out the case where $n - n_U > n_S$ and some eigenvalues are neither stable nor unstable, which is possible if $1 < r(\mathcal{E}_b)$.

4. The exogenous process Y_t is "behavioral-regular".

The first assumption rules out the problem of "decoupled" models (Sims, 2007) by ensuring that control variables can be mapped to the unstable block.[8] The second assumption rules out the inclusion of redundant equations that do not determine X_t (Sims, 2002). The third assumption generalizes one made by Klein (2000); under rational expectations, this only implies that the model has no unit roots. However, when the spectral radius is not one, this assumption potentially rules out models with generalized eigenvalues that are both stable and unstable.[9] The fourth assumption rules out rare edge cases where a forward-looking equation can be solved by both a control and a state variable. Blanchard and Kahn (1980) implicitly make this assumption in order for their condition to be necessary for existence; Appendix A defines the generalized "behavioral-regular" property and describes why it useful.

Definition 3 *A* ***stationary solution*** *is a stationary finite-variance time series X_t that is linear in current and past realizations of ω_t, and solves equation* (4) *for all t.*

Because the vector X_{t+1} may contain state variables that are known at time t, the expectations operator cannot be applied to it uniformly, so the Klein (2000) method is difficult to generalize directly. Instead, it is useful to follow the approach in Sims (2002), whereby X_{t+1} is divided into terms known at time t and forecast errors that are zero in expectation.

Let η_t denote the $n \times 1$ vector of forecast errors of X_t:

$$\eta_t \equiv \begin{pmatrix} 0 \\ C_t - \mathbb{E}^b_{t-1}[C_t] \end{pmatrix}$$

and let χ_t denote the $n \times 1$ vector of states and forecasts of controls:

$$\chi_t \equiv \begin{pmatrix} K_t \\ \mathbb{E}^b_t[C_{t+1}] \end{pmatrix}$$

[8]This rules out the auxiliary equation approach (Bianchi and Nicolò, 2021) to satisfy the Blanchard-Kahn condition in an indeterminate model.

[9]What happens if this assumption is violated? If eigenvalues are in the interval $[r(\mathcal{E}_b), 1]$ models can become internally inconsistent: variables may be solved recursively both forwards and backwards with conflicting outcomes. This does not necessarily invalidate a model, but additional *a priori* assumptions must assign the eigenvalues in $[r(\mathcal{E}_b), 1]$ as either stable or unstable. For example, Gabaix (2020) has eigenvalues in the interval $[r(\mathcal{E}_b), 1]$ and assigns them to be unstable, so that there are as many unstable eigenvalues as control variables.

A solution to the macroeconomic model satisfying Definition 3 is a pair of stationary processes η_t and χ_t satisfying equation (4), rewritten in terms of η_t and χ_t:

$$B_{X1}\chi_t = B_{X0}\chi_{t-1} + B_{X0}\eta_t + B_Y Y_t \tag{5}$$

such that η_t is the behavioral forecast error, i.e. forecasts must also satisfy the expectational equation:

$$B_{X1}\mathbb{E}^b_{t-1}[\chi_t] = B_{X0}\chi_{t-1} + B_Y\mathbb{E}^b_{t-1}[Y_t] \tag{6}$$

3.2 A Sufficient Condition for a Unique Solution

Condition 1 (Behavioral Blanchard-Kahn) *The numbers of non-predetermined control variables n_C, predetermined state variables n_K, unstable n_U and stable n_S generalized eigenvalues must satisfy*

$$n_C = n_U$$

and

$$n_K = n_S$$

This condition reduces to the sufficient conditions in Blanchard and Kahn (1980) and Klein (2000) under rational expectations, where the spectral radius is $r(\mathcal{E}) = 1$.

Theorem 1 *If a macroeconomic model satisfies the Behavioral Blanchard-Kahn condition, then it has a unique stationary solution.*

Proof. Apply the Schur decomposition to equations (5) and (6):

$$T_1 Z\chi_t = T_0 Z\chi_{t-1} + T_0 Z\eta_t + Q' B_Y Y_t \tag{7}$$

$$T_1 Z\mathbb{E}^b_{t-1}[\chi_t] = T_0 Z\chi_{t-1} + Q' B_Y \mathbb{E}^b_{t-1}[Y_t] \tag{8}$$

The first n_S dimensions of the operator equation (7) are the "stable block", while the remaining n_U dimensions are the "unstable block". To simplify these blocks, decompose the following vectors into the first n_S and the remaining n_U dimensions:

$$\begin{pmatrix} \chi^S_t \\ \chi^U_t \end{pmatrix} \equiv Z\chi_t \quad \begin{pmatrix} \eta^S_t \\ \eta^U_t \end{pmatrix} \equiv Z\eta_t \quad \begin{pmatrix} Y^S_t \\ Y^U_t \end{pmatrix} \equiv Q' B_Y Y_t \tag{9}$$

χ_t^U is determined by the unstable block of equation (8):

$$T_{1,U}\mathbb{E}_{t-1}^b[\chi_t^U] = T_{0,U}\chi_{t-1}^U + \mathbb{E}_{t-1}^b[Y_t^U] \tag{10}$$

Lemma 1 implies that this recursive equation has a unique fixed point χ_t^U, which is given by

$$\chi_t^U = -\sum_{j=0}^{\infty}\left(T_{0,U}^{-1}T_{1,U}\right)^j T_{0,U}^{-1}\mathbb{E}_{t,t+j}^b[Y_{t+j+1}^U] \tag{11}$$

With χ_t^U solved by equation (11), the unstable block of equation (7) implies a unique solution for η_t^U:

$$\eta_t^U = T_{0,U}^{-1}T_{1,U}\chi_t^U - \chi_{t-1}^U - T_{0,U}^{-1}Y_t^U \tag{12}$$

η_t^S is solved directly from η_t^U and the definitions $\begin{pmatrix} \eta_t^S \\ \eta_t^U \end{pmatrix} = Z\eta_t$, $Z = \begin{pmatrix} z_{SK} & z_{SC} \\ z_{UK} & z_{UC} \end{pmatrix}$, and $\eta_t = \begin{pmatrix} 0 \\ C_t - \mathbb{E}_{t-1}^b[C_t] \end{pmatrix}$:

$$\begin{pmatrix} \eta_t^S \\ \eta_t^U \end{pmatrix} = \begin{pmatrix} z_{SK} & z_{SC} \\ z_{UK} & z_{UC} \end{pmatrix}\begin{pmatrix} 0 \\ C_t - \mathbb{E}_{t-1}^b[C_t] \end{pmatrix}$$

$$\begin{pmatrix} \eta_t^S \\ \eta_t^U \end{pmatrix} = \begin{pmatrix} z_{SC}(C_t - \mathbb{E}_{t-1}^b[C_t]) \\ z_{UC}(C_t - \mathbb{E}_{t-1}^b[C_t]) \end{pmatrix}$$

z_{UC} is full rank, so $n_U = n_C$ implies that it is invertible and the forecast errors can be found by

$$C_t - \mathbb{E}_{t-1}^b[C_t] = z_{UC}^{-1}\tilde{\eta}_t^U$$

η_t^S is uniquely solved by

$$\eta_t^S = z_{SC}z_{UC}^{-1}\eta_t^U \tag{13}$$

Finally, solutions for χ_t^U, η_t^U, and η_t^S allow the remaining unknown χ_t^S to be solved from the stable block of equation (7):

$$T_{1,S}\chi_t^S + T_{1,SU}\chi_t^U = T_{0,S}\chi_{t-1}^S + T_{0,SC}\chi_{t-1}^U + T_{0,S}\eta_t^S + T_{0,SC}\eta_t^U + Y_t^S \tag{14}$$

rearrange for χ_t^S:

$$\chi_t^S = T_{1,S}^{-1} T_{0,S} \chi_{t-1}^S + \Theta_t \tag{15}$$

where

$$\Theta_t \equiv -T_{1,S}^{-1} T_{1,SU} \chi_t^U + T_{1,S}^{-1} T_{0,SC} \chi_{t-1}^U + T_{1,S}^{-1} T_{0,S} \eta_t^S + T_{1,S}^{-1} T_{0,SC} \eta_t^U + T_{1,S}^{-1} Y_t^S$$

The n_S smallest magnitude generalized eigenvalues are are also the eigenvalues of $T_{1,S}^{-1} T_{0,S}$. The Behavioral Blanchard-Kahn condition says that these eigenvalues are all stable, implying that $\left(I - T_{1,S}^{-1} T_{0,S} L\right)$ is invertible:

$$\chi_t^S = \left(I - T_{1,S}^{-1} T_{0,S} L\right)^{-1} \Theta_t \tag{16}$$

which is the unique solution for χ_t^S.

Z is unitary, so the time series X_t that uniquely solves the macroeconomic model (4) is recovered by

$$X_t = \chi_{t-1} + \eta_t = Z^* \begin{pmatrix} \chi_{t-1}^S \\ \chi_{t-1}^U \end{pmatrix} + Z^* \begin{pmatrix} \eta_t^S \\ \eta_t^U \end{pmatrix}$$

■

The proof of Theorem 1 is constructive; it proves a unique solution exists while also providing a method for calculating the solution.

3.3 Necessary Conditions for Uniqueness

The traditional Blanchard-Kahn condition is not just sufficient; it is necessary for a rational expectations solution to be unique. This is also true for the Behavioral Blanchard-Kahn condition if the model satisfies an additional property: Sunspot Admissibility. This section discusses this additional condition and introduces Theorems 2 and 3, which state precisely when model solutions are unique and when the Behavioral Blanchard-Kahn condition is necessary.

3.3.1 Sunspot Admissibility

The Sunspot Admissibility condition (introduced below) identifies when multiple equilibria are possible. This condition is redundant under rational expectations, but relevant when introducing behavioral expectations. The condition identifies when an underdetermined model ($n_C > n_U$) admits multiple equilibria. Sunspot Admissibility is always satisfied for rational expectations, where it is straightforward to construct sunspot equilibria. "Sunspots" are forecast errors, which can be any white noise process for rational expectations. But sunspots are nontrivial for many types of behavioral expectations; the forecast errors are rarely white noise, and can depend in complicated ways on the process for forecasts.

Some additional notation is necessary to introduce Sunspot Admissibility. Eigendecompose the matrix $T_{1,S}^{-1}T_{0,S} = Q_S^{-1}\Lambda_S Q_S$ so that Λ_S is the diagonal matrix of stable eigenvalues, including zeros. The regularity assumption implies that z_{UC} has rank n_U, so select from z_{SC} the $n_C - n_U$ rows $z_\odot$ such that $\begin{pmatrix} z_\odot \\ z_{UC} \end{pmatrix}$ is invertible. $\odot$ denotes the "sunspot" dimensions of X_t. The $(n_C - n_U) \times n_S$ matrix $s'_\odot$ extracts these dimensions from among the stable dimensions.

Condition 2 (Sunspot Admissibility) *There exists a bounded nonzero vector $\vec{v}$ such that*

$$s'_\odot Q_S^{-1}\left(\Lambda_S - \mathcal{E}_b\right)\left(I - \Lambda_S L\right)^{-1} Q_S s_\odot \vec{v} = 0$$

This condition is possible to check for a number of types of behavioral expectations without knowing anything else about the economic model. For example, under rational expectations, Sunspot Admissibility is always satisfied. For some heuristics, it is never satisfied. In other cases, Sunspot Admissibility is model dependent, but can be checked without first solving the model, by determining if the model is "simply stable."

3.3.2 Simple Stability

Definition 4 *A model is **simply stable** if there exists a square matrix φ such that any column of $Q_S s_\odot \varphi$ has a single non-zero entry.*

In some cases, this property is easy to check. If a model has at most one stable eigenvalue, then it necessarily is simply stable, because $Q_S s_\odot$ is a scalar. For example, this implies that the three-equation Behavioral New Keynesian model studied in Section 5 is simply stable for all types of expectations. More generally, simple stability is straightforward to check from the Schur decomposition.

Why is this property useful? Sunspot Admissibility is easily satisfied when the eigenvector of $\mathcal{E}_b$ associated with eigenvalue λ corresponds to an AR(1) process with autocorrelation λ. This is true in some cases (e.g. rational expectations) but typically an eigenvector of $\mathcal{E}_b$ is some process that depends on the eigenvalue in some other way. In these cases, it may be impossible to find a sufficient $\vec{v}$ if each sunspot dimension is associated with multiple distinct eigenvalues in Λ_S. However, if Q_S is simple enough so that at least one sunspot dimension can be associated with a single stable eigenvalue, then behavioral expectations with exotic eigenvectors can still satisfy Sunspot Admissibility. Proposition 1 formalizes this logic. Let $(Q_S s_\odot)_i$ indicate any column with only one nonzero entry, and let $\lambda_i^\odot$ denote its associated eigenvalue in Λ_S.

Proposition 1 *If there exists a $\lambda_i^\odot$ that is an eigenvalue of $\mathcal{E}_b$, then a simply stable model satisfies Sunspot Admissibility.*

Proof: Appendix B.4

Proposition 1 makes Sunspot Admissibility straightforward to evaluate even for the ambiguous cases, i.e. the types of expectations for which the condition neither always holds nor never holds. Evaluating whether the Sunspot Admissibility condition holds is valuable because it allows a practitioner to apply Theorem 2.

3.3.3 Sunspot Admissibility and Multiplicity

Theorem 2 *Consider a model with at least one solution. The model has multiple solutions if and only if $n_U < n_C$ and Sunspot Admissibility is satisfied.*

Proof: Appendix B.5

The proof is constructive; if any solution exists, it provides a method to derive any number of additional "sunspot" solutions. Theorem 2 is useful for economists who would like to determine exactly whether their solvable model has multiplicity. If Sunspot Admissibility fails to hold or $n_U \geq n_C$, then they are assured that there is no multiplicity. Otherwise, then there must be multiple solutions.

A corollary is the statement from the introduction:

Corollary 1 *A solution is unique if and only if the Behavioral Blanchard-Kahn or Sunspot Admissibility conditions hold.*

Proof: Appendix B.6

While Theorem 2 is practical when a model is known to have a solution, it does not tell whether a solution exists. The next Theorem 3 states when the Behavioral Blanchard-Kahn condition is necessary and sufficient for a unique solution to exist.

Theorem 3 *If a macroeconomic model satisfies the Sunspot Admissibility condition, then the Behavioral Blanchard-Kahn condition is necessary and sufficient for there to exist a unique solution.*

Proof. Sufficiency is given by Theorem 1.

The Behavioral Blanchard-Kahn condition fails if either $n_U \neq n_C$ or $n_S \neq n_K$. Using $n_U + n_S \leq n_C + n_K$, the four possible cases are:

1. If $n_U < n_C$, then there exist zero or multiple solutions, by Theorem 2.
2. If $n_S > n_K$, then it must be that $n_U < n_C$, so there is no unique solution.
3. If $n_S < n_K$, then there is no solution, by Lemma 2.
4. If $n_U > n_C$, then it must be that $n_S < n_K$, so there is no solution.

■

Table 1 summarizes the implications of the Behavioral Blanchard-Kahn and Sunspot Admissibility conditions studied in this section.

4 A Simple Asset Pricing Model

A simple example illustrates why the Behavioral Blanchard-Kahn condition is important, and how equilibrium uniqueness depends on the spectral radius $r(\mathcal{E}_b)$ of the expectation operator.

4.1 The Model

Consider an asset paying stochastic dividends d_t, which is governed by a stationary AR(1) process:

$$d_t = \rho d_{t-1} + \omega_t$$

where ω_t is standard normal.

	SSA holds	SSA fails
BBK holds	Unique solution	Unique solution
$n_S < n_K$	No solution	No solution
$n_U < n_C$ and $n_S \geq n_K$	Multiplicity	Unique solution

Table 1: Existence and Uniqueness Summary

Notes: The table summarizes when a model has a solution, and when it is unique, depending on the Behavioral Blanchard-Kahn (BBK) condition and the Sunspot Admissiblity (SSA) condition. n_S and n_U denote the number of stable and unstable generalized eigenvalues, while n_K and n_C denote the number of state and control variables. The results follow from Theorems 1, 2, 3, and Lemma 2.

The asset is priced by risk-neutral agents with fixed discount factor $\beta > 0$. The price of the asset once dividends are realized but before they are paid is given by

$$p_t = d_t + \beta \mathbb{E}_t^b \left[p_{t+1} \right] \tag{17}$$

When mapped to the general form in equation (4), $B_{X0} = 1$ and $B_{X1} = \beta$, so the generalized eigenvalue is $1/\beta$. When $\beta < 1$, there is one explosive eigenvalue and one endogenous control p_t, so the Blanchard-Kahn condition is satisfied under rational expectations.

The model solution is given by recurring equation (17):

$$p_t = d_t + \beta \mathbb{E}_t^b \left[d_{t+1} \right] + \beta^2 \mathbb{E}_t^b \left[\mathbb{E}_{t+1}^b \left[d_{t+2} \right] \right] + ... \tag{18}$$

if this sequence convergences. The operator representation of the equilibrium condition is

$$\vec{p} = \vec{d} + \beta \mathcal{E}_b \vec{p} \tag{19}$$

where

$$\vec{d} = \begin{pmatrix} 1 \\ \rho \\ \rho^2 \\ \vdots \end{pmatrix}$$

With operators, the solution is again given recursively by

$$\vec{p} = \vec{d} + \beta\mathcal{E}_b\vec{d} + \beta^2\mathcal{E}_b^2\vec{d} + ...$$

$$= (I - \beta\mathcal{E}_b)^{-1}\vec{d}$$

if it exists. By definition, the inverse $(I - \beta\mathcal{E}_b)^{-1}$ exists if the spectral radius satisfies $r(\mathcal{E}_b) < \frac{1}{\beta}$. This is the Behavioral Blanchard-Kahn condition for the simple asset pricing model, and it ensures the solution is unique.

4.2 Multiplicity

What goes wrong when $r(\mathcal{E}_b) > \frac{1}{\beta}$? Sunspot equilibria are possible, even though the traditional Blanchard-Kahn condition is satisfied.

Consider any two solutions $\vec{p}_1$ and $\vec{p}_2$ solving the asset pricing equation (19). The difference $\hat{p} \equiv \vec{p}_1 - \vec{p}_2$ satisfies

$$\hat{p} = \beta\mathcal{E}_b\hat{p} \tag{20}$$

Such a vector $\hat{p}$ only exists if $1/\beta$ is an eigenvalue of $\mathcal{E}_b$. This is only possible if Sunspot Admissibility is satisfied, which requires $r(\mathcal{E}_b) > 1/\beta$.

If so, then the asset pricing model features multiplicity. If $\vec{p}$ is a solution that depends on dividends d_t and $\hat{p}$ is an eigenseries driven by extrinsic sunspots, then $\vec{p} + \hat{p}$ is an equilibrium price process with excess volatility.

When the traditional Blanchard-Kahn condition fails in rational expectations models, sunspots are easy to construct: any white noise process can be a forecast error in the extrinsic process. But sunspots are not so trivial when agents have behavioral expectations.

As an example, consider the "overextrapolation" expectations studied by Angeletos, Huo, and Sastry (2021):

$$\mathbb{E}_t^{ME,\theta}\left[p_{t+1}\right] = \theta\mathbb{E}_t\left[p_{t+1}\right]$$

with $\theta > 1$. Section D.2.1 demonstrates that the spectral radius for this type of expectations is $r(\mathcal{E}_{ME,\theta}) = \theta$. If $\theta > \frac{1}{\beta}$, then the asset pricing model has multiple solutions. If $\theta < \frac{1}{\beta\rho}$, a forward-looking solution to equation (18) still exists:

$$p_t^{ME,\theta} = d_t + \beta \mathbb{E}_t^{ME,\theta}[d_{t+1}] + \beta^2 \mathbb{E}_t^{ME,\theta}[d_{t+2}] + ...$$

$$= \frac{1}{1 - \beta\theta\rho} d_t$$

With these expectations, a valid sunspot process satisfying equation (20) for $\hat{p}_t$ is

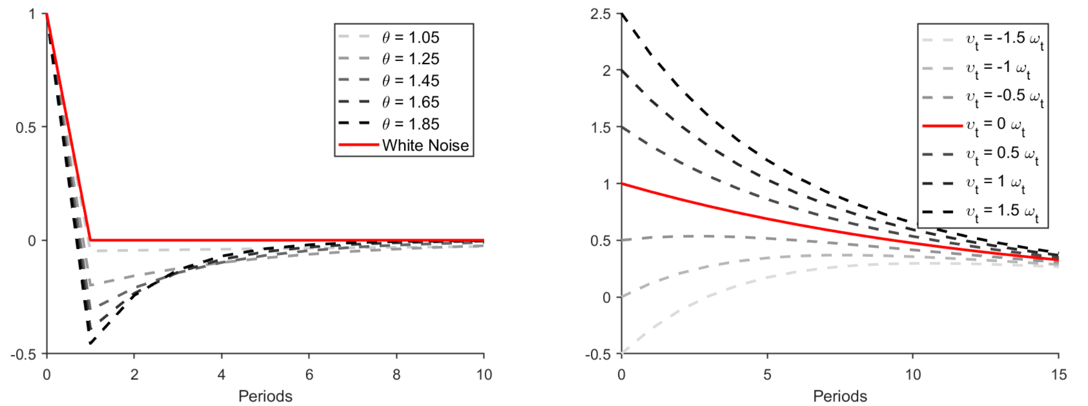
$$\hat{p}_t = \frac{1}{\beta\theta}\hat{p}_{t-1} + \upsilon_t \tag{21}$$

where υ_t is any white noise. The process for υ_t can be any scaling of the dividend shocks ω_t, or it can be extrinsic to the model, including literal sunspots. But from the perspective of agents in the model, forecast errors are not white noise. Instead, sunspot forecast errors $\hat{\eta}_t \equiv \hat{p}_t - \mathbb{E}_{t-1}^{ME,\theta}[\hat{p}_t]$ are ARMA(1,1):

$$\hat{\eta}_t = \frac{1 - \frac{1}{\beta}L}{1 - \frac{1}{\beta\theta}L}\upsilon_t$$

For some expectations, it is impossible to construct such a process; there are the expectations for which Sunspot Admissibility never holds. But for overextrapolation, an extrinsic eigenseries is admissible so long as $\theta > 1/\beta$.

Figure 2 displays multiple solutions to the asset pricing model for overextrapolation expectations with $\theta > 1/\beta$. Panel (a) plots how the sunspot forecast error depends on the behavioral parameter θ. Each line is the impulse response function of an equilibrium forecast error η_t to a unit sunspot shock υ_t. The solid red curve is white noise, which is the standard sunspot shock under rational expectations ($\theta = 1$). As θ increases, agents' forecasts become less rational, and the forecast errors look less like white noise. Panel (b) plots the impulse response functions for equilibrium prices p_t . In this panel, the behavioral parameter is fixed and the sunspot is defined as $\upsilon_t = \alpha\omega_t$ for different values of α. The solid red curve is the solution without any sunspots. All of these price paths are solutions to the model.



(a) Sunspot Forecast Errors by θ (b) Multiple Equilibrium Price Processes

Figure 2: Multiplicity in the Simple Asset Pricing Model

Notes: The impulse response functions (IRFs) are plotted for unit dividend shocks in the the asset pricing model with ME-type expectations, with $\beta = 0.99$ and $\rho = 0.75$ in all cases. The panel (a) IRFs are the responses of sunspot forecast errors, calculated for a variety of θ values; the "white noise" sunspots correspond to $\theta = 1$. The panel (b) IRFs are the responses of asset prices for different sunspot equilibria, with a common $\theta = 1.25$.

4.3 Sunspot Admissibility in the Asset Pricing Model

The examples considered thus far in this section satisfied Sunspot Admissibility: if the Behavioral Blanchard-Kahn condition did not hold, then there was multiplicity. But Theorem 2 says that if Sunspot Admissibility fails, then solutions to models will be unique regardless of the Behavioral Blanchard-Kahn condition.

The next two examples demonstrate why Sunspot Admissibility is necessary. In both cases, I assume $\beta > 1$ so that the Behavioral Blanchard-Kahn condition fails. Under rational expectations, this would imply multiplicity, this is not always the case when expectations are behavioral. Section 4.3.1 considers the "Naive Heuristic", which implies Sunspot Admissibility is never satisfied. Then Section D.3.4 considers "Natural Expectations", for which the condition holds in some models but not others.

4.3.1 Asset Pricing with the Naive Heuristic

Suppose now that asset-pricing agents forecast with the "Naive Heuristic" (Brock and Hommes, 1997). Agents with NH-type expectations forecast prices using the current price: $\mathbb{E}_t^{NH}[p_{t+1}] = p_t$. How are assets priced with these expectations? The equilibrium condition (17) becomes

$$p_t = d_t + \beta \mathbb{E}_t^{NH}[p_{t+1}]$$

$$\implies p_t = \frac{d_t}{1-\beta}$$

This solution holds for any value of $\beta \neq 1$. Unlike the FIRE case, a sufficiently small β is not required for a stationary equilibrium to exist. And this equilibrium is always unique.

When $\beta > 1$, the Behavioral Blanchard-Kahn condition fails, but that is not enough to guarantee multiplicity. For a sunspot equilibrium to exist, there must be a time series for equilibrium differences $\hat{p}_t$ that satisfies equation (20). With the Naive Heuristic, this equation becomes:

$$\hat{p}_t = \beta \hat{p}_t$$

For $\beta > 1$, the only solution is $\hat{p}_t = 0$. No multiplicity is possible.

The key characteristic that makes Sunspot Admissibility fail is that the heuristic is entirely backwards-looking. Property 3 implies that these types of heuristics never admit sunspots.

For the expectations discussed thus far, Sunspot Admissibility either always held (such as the ME-type) or never held (such as the NH-type). But there are expectations that fall in between, such that sunspots are possible in some models but not others. The next section provides an example

4.3.2 Asset Pricing with Natural Expectations

The previous section demonstrated why multiplicity requires Sunspot Admissibility. With the Naive Heuristic, this condition was straightforward to characterize: it never holds. But for some expectations, it is not so clear, and determining if a model is *simply stable* can help a practitioner evaluate whether multiplicity is possible.

In this example, agents have "Natural Expectations."[10] They forecast prices by $\mathbb{E}_t^{NE,\theta,\phi}[p_{t+1}] = (1-\phi)\mathbb{E}_t[p_{t+1}] + \phi\theta p_t$, with $\phi \in (0,1)$ and $\theta \in (-1,1)$. In this case,

[10]These expectations are inspired by – but do not exactly correspond to – the expectations studied in Fuster, Laibson, and Mendel (2010).

the simple asset pricing equation (17) becomes:

$$p_t = d_t + \beta\left((1-\phi)\mathbb{E}_t[p_{t+1}] + \phi\theta p_t\right)$$

Table 2 reports that the Behavioral Blanchard-Kahn condition is satisfied in this model whenever $1/\beta > \max(|\phi\theta|, 1-\phi)$.

What happens when β is large enough so that the condition is not satisfied? Multiplicity is possible. To form sunspot equilibria, there must be a time series $\hat{p}_t$ that satisfies equation (20); with natural expectations, this equation becomes:

$$\hat{p}_t = \beta\left((1-\phi)\mathbb{E}_t[\hat{p}_{t+1}] + \phi\theta\hat{p}_t\right)$$

These expectations have an AR(1) eigenseries, i.e. this equation is solved by the following time series:

$$\hat{p}_t = \frac{1/\beta - \phi\theta}{1-\phi}\hat{p}_{t-1} + \upsilon_t$$

where υ_t is any white noise.

Sunspot Admissibility is satisfied in this case if and only if $|\frac{1/\beta-\phi\theta}{1-\phi}| < 1$. Otherwise, it is impossible to create a stationary sunspot process for prices. This condition is known analytically because the simple asset pricing model is *simply stable*: there is only one stable eigenvalue for the one sunspot dimension. To see what happens without this property, consider the following *modified* asset pricing model:

$$p_t = d_t - \alpha\beta x_{t-1} + \beta\mathbb{E}_t^{NE}[p_{t+1}] \tag{22}$$

where x_t is some endogenous state variable that affects the net dividends earned by the asset pricer. The state variable is determined based on past prices:

$$x_t = \gamma p_t + \delta x_{t-1} \tag{23}$$

where $\delta \in (0,1)$ by assumption. This example is rather abstract, so as to be only a simple modification of the asset pricing model, and to have its stability easily describable. But this x_t state might represent a dynamic tax or other market intervention that depends on asset price levels in a sticky way.

Represent this model in the form of the matrix equation (4):

$$\mathbb{E}_t^{NE}\left[\begin{pmatrix} p_{t+1} \\ x_t \end{pmatrix}\right] = \begin{pmatrix} \eta & \alpha \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} p_t \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} d_t$$

where $\eta \equiv 1/\beta$.

The proof of Theorem 2 is constructive and uses a a general operator representation to show why sunspot admissibility is necessary for multiplicity. But to make it clear what can go wrong, I will use a simpler approach to demonstrate why sunspots may not be possible in this particular model. Substitute out for from the asset pricing equation (22) using $x_t = \frac{\gamma}{1-\delta L} p_t$ as implied by equation (23). This gives a single equation that describes equilibrium:

$$\left(\eta + \frac{\alpha\gamma L}{1-\delta L}\right) p_t = d_t + \mathbb{E}_t^{NE}[p_{t+1}]$$

If the Behavioral Blanchard-Kahn condition is not satisfied, will there be sunspot equilibria? Rewrite this single equation in terms of differences between equilibrium price processes $\hat{p}_t$ (e.g. $\hat{p} = \vec{p}_1 - \vec{p}_2$, so the exogenous d_t drops out, analogous to equation (20)), and use the definition of the NE expectation:

$$\left(\eta + \frac{\alpha\gamma L}{1-\delta L}\right) \hat{p}_t = (1-\phi)\mathbb{E}_t[\hat{p}_{t+1}] + \phi\theta\hat{p}_t \tag{24}$$

Does this equation have a stationary solution? Not in general. Proposition 2 gives the condition for exactly when. Denote the determinant of the model's matrix by $\zeta \equiv \delta\eta - \alpha\gamma$:

Proposition 2 *The modified asset pricing model with natural expectations has stationary sunspot equilibria if and only if both eigenvalues are stable,* $|\zeta - \delta\phi\theta| < 1 - \phi$, *and* $|\phi\theta - \eta - \delta(1-\phi)| < 1 - \phi + \zeta - \delta\phi\theta$.

Proof: Appendix B.7

Crucially, the condition for Proposition 2 may or may not be satisfied regardless of whether $|\frac{1/\beta - \phi\theta}{1-\phi}| < 1$ holds. The analytical condition is useful for simply stable models, but when models are more complicated, there is no general analytical inequality that determines whether Sunspot Admissibility holds.

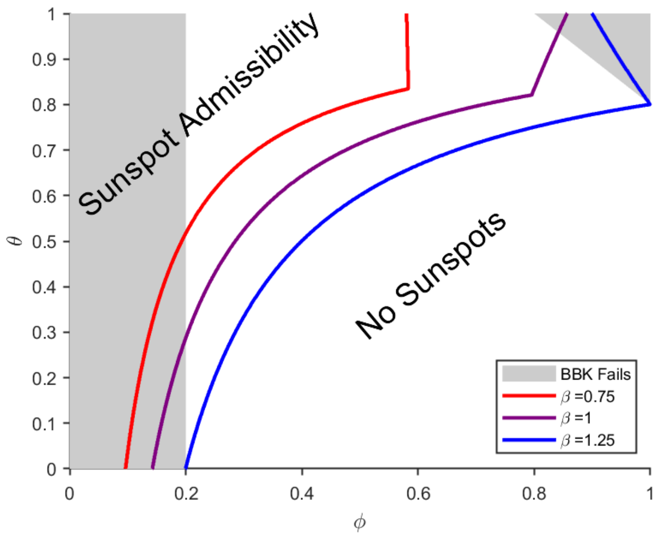


Figure 3: Multiplicity in the Simple Asset Pricing Model

Notes: The figure demarcates the regions where Sunspot Admissibility and the Behavioral Blanchard-Kahn conditions hold in the modified asset pricing model for possible values of ϕ, θ, and β. In all cases the eigenvalues are $\lambda_1 = \frac{4}{5}$ and $\lambda_2 = \frac{1}{2}$. The remaining δ, η, and ζ are determined by the values of λ_1, λ_2, and β

Figure 3 demonstrates this challenge. The gray regions denote combinations of the expectation parameters ϕ and θ such that the Behavioral Blanchard-Kahn condition fails when the largest eigenvalue is $\lambda_1 = \frac{4}{5}$. The blue curve demarcates the regions where Sunspot Admissibility is or is not satisfied in the simple asset pricing model. In the gray regions left of the curve, multiplicity is possible because both Behavioral Blanchard-Kahn fails and Sunspot Admissibility is satisfied. However, in the gray region to the right of the curve, there can be no sunspot equilibria even though Behavioral Blanchard-Kahn fails; this is the space where $|\frac{1/\beta - \phi\theta}{1-\phi}| > 1$.

Modifying the model to break simple stability changes the relationship between expectations and multiplicity. The three curves in Figure 3 plot the demarcations for three different β parameterizations that hold the eigenvalues $\lambda_1 > \lambda_2 > 0$ fixed. The blue line that describes the simple model also corresponds to the modified model where $\alpha = \gamma = 0$, and $\delta = \lambda_2$. The purple and red lines have smaller values of β but the same eigenvalues, so α and γ must become non-zero: the model's control and state now interact. The model is no longer simply stable, so the determinacy region changes. As β shrinks, the curves shift left, reducing the possibility of multiplicity by shrinking the space where both Behavioral Blanchard-Kahn and Sunspot Admissibility are satisfied.

This example demonstrates why simple stability is a useful property for many expectations: it allows for an analytical inequality that determines exactly when Sunspot Admissibility holds. But simple stability is not a necessary condition; if it fails to hold, Sunspot Admissibility may have to be evaluated case-by-case.

5 The New Keynesian Model with Behavioral Expectations

This section considers the three-equation New Keynesian model with general behavioral expectations, and describes expectations for which unique equilibria exist when nominal interest rates are fixed.

5.1 The Behavioral New Keynesian Model

The three-equation New Keynesian model is given by:

New Keynesian Phillips curve:	$\pi_t = \beta \mathbb{E}_t^b[\pi_{t+1}] + \kappa y_t$
Euler equation:	$i_t = \mathbb{E}_t^b[\pi_{t+1}] + \mathbb{E}_t^b[\gamma y_{t+1}] - \gamma y_t$
Taylor rule:	$i_t = \phi_\pi \pi_t + x_t$

where the endogenous variables are per capita output y_t, the inflation rate π_t, and the nominal interest rate i_t. x_t is an exogenous monetary policy shock. In the canonical model, $\mathbb{E}^b$ denotes rational expectations. Under some assumptions (Appendix C) an arbitrary behavioral expectations operator can be used instead; Jump and Levine (2019) review several specific examples.

When $\phi_\pi < 1$, this model is well known to not satisfy the Blanchard-Kahn conditions for standard calibrations: there are three control variables, but only two explosive eigenvalues. The "Taylor principle" is to resolve this multiplicity by choosing $\phi_\pi > 1$ so that there are three explosive eigenvalues. Per Theorem 1, this principle applies for any behavioral expectation with $r(\mathcal{E}_b) = 1$.[11]

[11] This includes the New Keynesian models with diagnostic expectations studied by Bianchi, Ilut, and Saijo (2024) and L'Huillier, Singh, and Yoo (2023).

5.2 Determinacy with an Interest Rate Peg

Recent experience conflicts with the canonical model. Rich economies spent several years with fixed interest rates at the zero lower bound after the global financial crisis, and yet inflation volatility remained low. A modest literature arose to suggest mechanism under which the New Keyensian model is determined with an interest rate peg, including by relaxing rational expectations.

As demonstrated in Section 4.2, Gabaix (2020)'s "cognitive discounting" can achieve determinacy by choosing expectations with spectral radius $r(\mathcal{E}_{ME})$ less than the magnitude of the stable eigenvalue λ_S.[12] This choice ensures that the Behavioral Blanchard-Kahn condition is satisfied so a unique equilibrium must exist.

Similar to Gabaix's assumption, any form of behavioral expectations with spectral radius $r(\mathcal{E}_b) < |\lambda_S|$ will have a unique solution. With appropriate parameterization, this includes all of the incomplete information structures discussed in Section 7.[13]

Other forms of behavioral expectations can achieve determinacy without assuming biases so large that $r(\mathcal{E}_b) < |\lambda_S|$. This is possible if the expectations do not satisfy Sunspot Admissibility. Examples in this paper include many of the heuristic expectations described in Section 7. The most well-known example is Adaptive Expectations, which never admits any sunspot equilibria, even when the Behavioral Blanchard-Kahn condition is violated.[14] Proposition 3 makes this clear:

Proposition 3 *If the behavioral expectation is strictly backwards-looking, then the solution to the behavioral New Keynesian model with an interest rate peg is unique.*

Proof: Appendix B.8

When the behavioral expectation is backwards looking, it is impossible to construct sunspot forecast errors. The expectation operator has no eigenseries. Table 2

[12] Ilabaca, Meggiorini, and Milani (2020) evaluate the cognitive discounting model with US time series. Their parameter estimates satisfy $r(\mathcal{E}_{ME}) < 1$ so that the model is determinate in the pre-1980 data, even though monetary policy in this era responds less than one-for-one to inflation.

[13] Models that achieve determinacy in this way violate regularity condition (3): there exists an eigenvalue in the interval $(r(\mathcal{E}_b), 1]$. This violation is not problematic in the canonical New Keynesian model, because it has no state variables. However, these types of expectations can introduce internal inconsistencies in more general models where state variables may be solved both forwards and backwards.

[14] This result depends on the restriction of solutions to be stationary; McCallum (1983) demonstrates that adaptive expectations admit non-stationary sunspots equilibria. Of course, the rejection of nonstationary equilibria also motivates the Taylor principle, which achieves uniqueness by transforming the multiplicity of stationary equilibria into explosive solutions.

reports that many common heuristics in the literature are strictly backwards looking and imply a unique solution with an interest rate peg.

6 Sophistication vs. Naivete

Behavioral models may have different equilibria when agents are *sophisticated* than when they are *naive*, using the O'Donoghue and Rabin (1999) terminology.

The macroeconomic models considered in this paper feature "sophisticated" agents: they form expectations about future endogenous variables that are consistent with behavioral expectations holding in the future. This assumption is implicit in how the solution is defined (Definition 3); the time series that agents forecast in the model equation (4) are the equilibrium time series.

The simple asset pricing model of Section 4 makes it clear how the recursive model definition implies that agents are sophisticated. Recurring equation (17) implies equation (18): agents forecast the asset price p_{t+1} assuming that expectations will still be behavioral in $t+1$. But despite the terminology, the agents do not require any sophisticated reasoning in this model. At time t they are simply forecasting the stationary time series p_{t+1}; they do not need to reason through the model equilibrium.

In contrast, some papers feature "naive" agents: they forecast future variables expecting rational expectations to hold in the future. These models are solved by recurring equation (4) under rational expectations, and then applying the behavioral expectation once. In the simple asset pricing mode, the naive analog of equation (18) is:

$$\text{[naive:]} \ \ p_{N,t} = d_t + \beta \mathbb{E}_t^b \left[d_{t+1} + \beta^2 d_{t+2} + \beta^3 d_{t+3} + ... \right]$$

The recursive asset pricing equation (17) does not necessarily hold when expectations are naive. There are some examples where the naive solution and sophisticated solution coincide – i.e. for types where the law of iterated expectations holds, included Delayed Observation and Diagnostic Expectations with only 1-period of overreaction – but in general, the solutions may be different.

Some authors argue that *a priori*, the naive representation is a better description of human behavior.[15] But other macroeconomists assume agents are sophisticated,

[15] Bianchi, Ilut, and Saijo (2024) argue that naivety is more realistic, particularly when agents forecast their own behavior, and use the assumption to study business cycle models with diagnostic expectations.

including Gabaix (2020). The correct assumption may depend on the specific application.

If Theorems 1 and 2 apply to models with sophisticated agents, what determines existence and uniqueness when agents are naive? This problem is much simpler: *when naive agents have behavioral expectations, then the traditional Blanchard-Kahn condition can be applied as if agents had rational expectations.* Theorem 4 formalizes this result.

Let variables with N subscripts denote solutions to the model with naive agents. A "stationary naive solution" is a stationary time series $X_{N,t}$ satisfying

$$B_{X1}\chi_{N,t} = B_{X0}X_{N,t} + B_Y Y_t \tag{25}$$

for all t, where $\chi_{N,t}$ denotes the naive forecast of $X_{N,t+1}$. With this definition, the existence and uniqueness theorem is

Theorem 4 *If a macroeconomic model with naive agents has exactly n_C eigenvalues outside the unit circle, then it has a unique stationary naive solution.*

Proof: Appendix B.9

The proof of Theorem 4 closely resembles the proof of Theorem 1. Intuitively, the condition for a unique equilibrium with naive agents is the same as the condition for a unique equilibrium with rational expectations. Why? Naive agents' choices depend on their forecast of a counterfactual rational expectations economy. So if the counterfactual equilibrium is unique, their choices will be uniquely determined, no matter how their behavioral expectation is formed.

7 The Spectra for Various Expectations

In this section, I describe the spectra of a variety of expectations.

Table 2 summarizes the expectations. Section 7.2 introduces properties of operators that are useful for describing their spectra. Appendix D describes in greater detail how the operator forms and spectral properties are derived.

Figure 1 classified expectations into one of four categories, based on equilibrium uniqueness when the original Blanchard and Kahn (1980) condition holds. This classification is a function of the properties reported in in Table 2. Specifically, Theorem 2 implies that expectations are classified based on whether their properties satisfy:

Expectation Type	Operator Notation	Spectral Radius	When is Sunspot Admissibility Satisfied?
Rational Expectations	$\mathcal{E} = L'$	1	Always
Sub-rational Expectations			
Mis-extrapolation	$\mathcal{E}_{ME,\theta} = \theta\mathcal{E}$	θ	Simply Stable Models with $\lvert\lambda^{\odot}\rvert < \theta$
Diagnostic Expectations	$\mathcal{E}_{D,\theta,\phi} = \mathcal{E} + \theta(1 - L^{\phi}\mathcal{E}^{\phi})\mathcal{E}$	1	All Simply Stable Models
Delayed Observation	$\mathcal{E}_{DO} = L\mathcal{E}^2$	1	Always
Sticky Information	$\mathcal{E}_{SI,\theta} = \theta\sum_{j=0}^{\infty}(1-\theta)^j L^j \mathcal{E}^{j+1}$	1	All Simply Stable Models
Heuristic Expectations			
Adaptive Expectations	$\mathcal{E}_{AE,\theta} = \frac{\theta}{1-(1-\theta)L}$	θ	Never
Naive Heuristic	$\mathcal{E}_{NH,\theta} = L^{\theta}$	1	Never
Behavioral Learning	$\mathcal{E}_{BL,\theta} = \theta$	θ	Never
Natural Expectations	$\mathcal{E}_{NE,\theta,\phi} = \phi\theta + (1-\phi)\mathcal{E}$	$\max(\lvert\theta\phi\rvert, 1-\phi)$	Simply Stable Models with $\lvert\frac{\lambda^{\odot}-\theta\phi}{1-\phi}\rvert < 1$
Trend Following	$\mathcal{E}_{TF,\theta} = (1+\theta)L - \theta L^2$	$1+\theta$	Never
Anchoring and Adjustment	$\mathcal{E}_{AA,\theta} = (1+\theta)L - L^2$	$1+\theta$	Never
Heterogeneous Expectations	$\mathcal{E}_{HEH,\theta,\phi} = (1-\phi)\mathcal{E} + \phi\theta L$	$\max(\phi, \lvert(1-\phi)\theta\rvert)$	S.S. Models with $\lvert\theta\rvert < \frac{\phi}{1-\phi}$ and $\lvert\lambda^{\odot}\rvert < \phi + (1-\phi\theta)$
Incomplete Information			
Noisy Signals	$\mathcal{E}_{NS,\theta} = \frac{1}{1+\theta}\mathcal{E}$	$\frac{1}{1+\theta}$	Simply Stable Models with $\lvert\lambda^{\odot}\rvert < \frac{1}{1+\theta}$
Beauty Contests	$\mathcal{E}_{BC,\theta,\phi} = \frac{\theta+\phi}{\theta+\phi+\theta\phi}\mathcal{E}$	$\frac{\theta+\phi}{\theta+\phi+\theta\phi}$	Simply Stable Models with $\lvert\lambda^{\odot}\rvert < \lvert\frac{\theta+\phi}{\theta+\phi+\theta\phi}\rvert$
Signal Precision Overconfidence	$\mathcal{E}_{SPO,\theta,\phi} = \frac{\phi}{\phi+\theta}\mathcal{E}$	$\frac{\phi}{\phi+\theta}$	Simply Stable Models with $\lvert\lambda^{\odot}\rvert < \frac{\phi}{\theta+\phi}$

Table 2: Types of Expectations

Notes: The parameters θ and ϕ are defined differently for each type of expectations. Eigenvalues with magnitude larger than the spectral radius are classified as "unstable" for the purposes of Theorem 1. When the radius is one, the Blanchard-Kahn condition of the rational expectations model implies a unique solution exists in the behavioral expectations model. $\lambda^{\odot}$ denotes any stable eigenvalue associated with a sunspot dimension.

1. **Unique if Traditional Blanchard-Kahn Holds:** Spectral radius $r(\mathcal{E}_b) = 1$ and Sunspot Admissibility always satisfied

2. **Possible Multiplicity if Traditional Blanchard-Kahn Holds:** Spectral radius $r(\mathcal{E}_b) > 1$ and Sunspot Admissibility may be satisfied

3. **Possibly Unique if Traditional Blanchard-Kahn Fails:** Spectral radius $r(\mathcal{E}_b) < 1$ or Sunspot Admissibility may not be satisfied

4. **Always Unique:** Sunspot Admissibility never satisfied

7.1 Types of Expectations

Appendix D describes the expectations in detail, but I summarize them here. The behavioral expectations are classified into three categories.

Sub-rational Expectations are rational at long horizons, but not in the short run. These forms tend to satisfy Sunspot Admissibility, at least in simply stable models. "Mis-extrapolation" includes overextrapolation (Angeletos, Huo, and Sastry, 2021) and cognitive discounting (Gabaix, 2020) representations. "Delayed Observation" are expectations formed when agents forecast rationally but do not have access to contemporaneous information. Other expectations in this category include "Diagnostic Expectations" (Bordalo, Gennaioli, and Shleifer, 2018) and "Sticky Information" (Mankiw and Reis, 2002).

Heuristic Expectations are at least partially backwards-looking. When they are entirely backwards looking, they never satisfy Sunspot Admissibility, as with Adaptive Expectations (Cagan, 1956), the "Naive Heuristic" (Brock and Hommes, 1997), "Trend Following" and "Anchoring and Adjustment" (Hommes, Massaro, and Weber, 2019). Some forms use incorrectly-specified but rationally estimated AR(1) models, e.g. "Behavioral Learning" (Hommes and Zhu, 2014) or "Natural Expectations" (Fuster, Laibson, and Mendel, 2010). These forms do not have a linear representation, but I approximate them with a heuristic AR(1) instead. Finally, the expectations that mix forward-looking behavior with a heuristic admit sunspots in some cases, such as the "Heterogeneous Expectations" (Branch and McGough, 2004).

Incomplete Information includes simple dispersed information structures, where agents have rational expectations but noisy signals about economic shocks. The resulting expectation operator describes the behavior of average expectations. This is distorted when agents engage in "Beauty Contests" (e.g. Woodford (2003)) or are overconfident about their signal precisions (e.g. Daniel, Hirshleifer, and Subrahmanyam (1998) and Odean (1998).)

7.2 Spectral Properties of Expectations Operators

The spectral radius has several well-known properties that are useful for characterizing behavioral expectations:

1. The spectral radius is absolutely homogeneous, i.e. for scalar α:

$$r(\alpha\mathcal{E}_b) = |\alpha| r(\mathcal{E}_b)$$

2. If $\mathcal{E}_b$ is Toeplitz, then the spectral radius is the absolute value of the largest entry in the operator.

3. The spectral radius obeys Gelfand's formula:

$$r(\mathcal{E}_b) = \lim_{j\to\infty} ||\mathcal{E}_b^j||^{\frac{1}{j}} \tag{26}$$

   where $||\cdot||$ denotes the operator norm.

These properties yield reliable strategies for finding the spectral radius once an expectation's operator form is known. If the expectation is proportional to another operator with a known spectral radius, then the first property gives the radius immediately. If the operator is Toeplitz, the second property is useful. Finally, if none of those cases apply to the operator, Gelfand's formula can always be used, either analytically or numerically.

In order to describe when expectations satisfy the Sunspot Admissibility condition, it is necessary to study their eigenvalues and associated eigenvectors. A useful property to this end is:

Property 3 *Lower triangular Toeplitz operators have no eigenvalues.*

Altun (2011) proves this property for infinite dimensional Toeplitz operators. Many heuristic expectations have lower triangular operators, so they will never admit sunspot equilibria.

How can the eigenvalues of an expectation operator inform when the Sunspot Admissibility Condition is satisfied? Every eigenvalue λ of $\mathcal{E}_b$ has an associated stationary eigenseries y_t (representing the eigenvector of $\mathcal{E}_b$) satisfying

$$\mathbb{E}_t^b[y_{t+1}] = \lambda y_t$$

If y_t is AR(1) with autocorrelation λ_b, and the set of eigenvalues is the unit disc, then Sunspot Admissibility is always satisfied with white noise sunspots, as discussed below. If the eigenseries exists in some other form, then it is satisfied in simply stable models with stable eigenvalues that are also eigenvalues of the expectation operator. Otherwise, if y_t does not exist, the condition is never satisfied

8 Conclusion

This paper developed a general framework for representing behavioral expectations in macroeocnomic models, and studied equilibrium uniqueness therein. I introduced the Behavioral Blanchard-Kahn condition, which allows theorists and applied macroeconomists to understand how their assumptions about expectation formation affects multiplicity in their model. The condition depends on the spectral radius of the expectation operator. In the appendices, I derive the spectral radius for many common examples.

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A Behavioral-Regular Time Series

One of the assumptions made in Section 3.1 is that a model's time series is "behavioral-regular". This section defines this property, describes some features of time series with this property, and gives examples of what can go wrong when it does not hold.

In most practical cases, the exogenous time series Y_t is likely to be behavioral-regular. The assumption that Y_t satisfies the property rules out rare edge cases so that Theorem 3 can make a sharp if-and-only-if statement.

This requirement for eliminating edge cases is not specific to behavioral expectations. Blanchard and Kahn (1980) also implicitly assert this condition when they conclude that an overdetermined model "almost always has no solution." The behavioral-regular assumption makes this elimination explicit. However, when agents have general behavioral expectations, the condition is more complicated than the simple rational expectations case.

A.1 Definition

Why is this regularity property "behavioral"? The exact condition depends on the type of behavioral expectations in the model:

Definition 5 *The exogenous time series Y_t in a model is* ***behavioral-regular*** *if it satisfies both:*

1. $M_{UC}\Xi_{\eta,U}\vec{Y}^U \neq 0$
2. $\sum_{j=0}^{\infty}\left(T_{1,S}^{-1}T_{0,S}L\right)^j\left(\Upsilon\vec{Y}^C + T_{1,S}^{-1}\vec{Y}^S\right)$ *is unbounded if any eigenvalue of $T_{1,S}^{-1}T_{0,S}$ is outside the unit disk.*

where M_{UC} denotes the projection matrix given by

$$M_{UC} \equiv I - z_{UC}(z_{UC}'z_{UC})^{-1}z_{UC}'$$

$\Xi_{\eta,U}$ is the operator given by

$$\Xi_{\eta,U} = -\left(I - L\mathcal{E}_b\right)\left(I - T_{0,U}^{-1}T_{1,U}\mathcal{E}_b\right)^{-1}T_{0,U}^{-1}$$

Υ is the operator given by

$$\Upsilon \equiv T_{1,S}^{-1}\left(T_{1,SU} - (T_{0,SC}L)\left(I - T_{0,U}^{-1}T_{1,U}\mathcal{E}_b\right)^{-1}T_{0,U}^{-1}\mathcal{E}_b + \left(T_{0,S}z_{SC}z_{UC}^{-1} + T_{0,SC}\right)\Xi_{\eta,C}\right)$$

$\vec{Y}^S$ and $\vec{Y}^U$ are the partitioned block of $Q'B_Y\vec{Y}$ defined in equation (9).

Property (1) implies that the n_U-dimensional unstable block of the model requires n_U controls to solve, i.e. forward-looking equations cannot be solved by state variables. Property (2) implies that state variables cannot solve backwards-looking equations unless the associated eigenvalues are stable. Under rational expectations, this second property is redundant. But behavioral expectations allows for eigenvalues in the $(1, r(\mathcal{E}_b))$ interval that are neither stable nor unstable.

A.2 Implications of Behavioral-Regularity

Lemma 2 *If a model has $n_S < n_K$, and Y_t is behavioral-regular, then the model has no solution.*

Proof. First consider the case where $n_U > n_C$. η_t^U is related to forecast errors by

$$\eta_t^U = z_{UC}\left(C_t - \mathbb{E}_{t-1}^b[C_t]\right)$$

z_{UC} has more rows than columns, so the forecast error $C_t - \mathbb{E}_{t-1}^b[C_t]$ may be overdetermined. Suppose a solution for η_t^U exists. Left multiply by M_{UC}:

$$M_{UC}\eta_t^U = 0$$

because $M_{UC}z_{UC} = 0$. In operator notation $\vec{\eta}^U = \Xi_{\eta,U}\vec{Y}^U$:

$$M_{UC}\Xi_{\eta,U}\vec{Y}^U = 0$$

which violates the assumption that Y_t is behavioral-regular. No solution exists.

Next consider the case where $n_U \leq n_C$ so that solutions for χ_t^U, η_t^U, and η_t^S exist by equations (11), (12), and (13). A solution must have a square-summable process χ_t^S that satisfies recursive equation (15), which implies $\chi_t^S = \sum_{j=0}^{\infty}\left(T_{1,S}^{-1}T_{0,S}L\right)^j\Theta_t$.

In operator notation, this equation is

$$\vec{\chi}^S = \sum_{j=0}^{\infty} \left(T_{1,S}^{-1} T_{0,S} L\right)^j \left(\Upsilon \vec{Y}^C + T_{1,S}^{-1} \vec{Y}^S\right)$$

which is not square-summable if Y_t is behavioral-regular. ■

A.2.1 Problem Cases Without Behavioral-Regularity

What can go wrong when Y_t is not behavioral-regular?

First, consider the simple asset pricing model form Section 4. But instead of letting the price p_t be a control variable, assume that it is a pre-determiend state, denoted by p_t^S. The asset pricing equation (17) becomes:

$$p_{t-1}^S = d_t + \beta p_t^S$$

with $\beta \in (0, 1)$. Rearrange:

$$p_t^S = -\lambda \left(\sum_{j=0}^{\infty} \lambda^j L^j\right) d_t$$

where the model's eigenvalue $\lambda = \frac{1}{\beta} > 1$. This expression is "almost always" infinite, unless β is a zero of the dividend process, i.e.

$$d_t = (1 - \lambda L) x_t$$

where x_t is any square-summable time series. Only in this case, the state variable price p_t^S solves the model by

$$p_t^S = -\lambda x_t$$

This is an example of an edge case where $n_U > n_C$ but the model still has a solution because a state variable can solve the forward-looking asset pricing equation. The exogenous time series Y_t is the dividend process d_t, and it is not behavioral-regular because property (2) in definition 5 does not hold.

This example also fails property (1) in the rational expectations case. Suppose as usual that the price p_t is a control variable, and let the dividend process be MA(1):

$d_t = (1 - \lambda L)v_t$. The asset price is

$$p_t = d_t + \beta \mathbb{E}_t[d_{t+1}]$$

$$= v_t - \lambda v_{t-1} - \beta \lambda v_t$$

so in the edge case where $\lambda = \frac{1}{\beta}$:

$$p_t = -\lambda v_{t-1}$$

and again, the forward-looking pricing equation is solved by a predetermined state variable.

But when agents have behavioral expectations, $\lambda = \frac{1}{\beta}$ does not imply that property (1) fails. This is why definition 5 requires two properties, where the rational expectations analog would only require one. As an example, consider the ME-type expectations $\mathbb{E}_t^{ME,\theta}[p_{t+1}] = \theta \mathbb{E}_t[p_{t+1}]$. The asset pricing equation becomes

$$p_t = d_t + \beta\theta \mathbb{E}_t[d_{t+1}]$$

again, assume MA(1) dividends:

$$= v_t - \lambda v_{t-1} - \beta\theta\lambda v_t$$

Now with these expectations, property (1) fails if $\lambda = \frac{1}{\beta\theta}$.

B Additional Proofs

B.1 Proof of Property 2

Proof. By assumption $\mathcal{E}_b$ is series-agnostic, so it commutes with B and the spectral radius of $B\mathcal{E}_b$ is sub-multiplicative:

$$r(B\mathcal{E}_b) \leq r(B)r(\mathcal{E}_b) = |\lambda_B| r(\mathcal{E}_b) < 1$$

where the final inequality follows from the assumption that $|\lambda_B| < r(\mathcal{E}_b)^{-1}$. From the definition of the spectral radius, $\lambda I - B\mathcal{E}_b$ is invertible for any $\lambda > r(B\mathcal{E}_b)$. Therefore

$r(B\mathcal{E}_b) < 1$ implies $I - B\mathcal{E}_b$ is invertible. ■

B.2 Proof of Lemma 1

Proof. Expressed in operator notation, the recursive equation is

$$\vec{x} = B\mathcal{E}_b\vec{x} + \mathcal{E}_b\vec{y}$$

Property 2 implies $I - B\mathcal{E}_b$ is invertible, so $\vec{x}$ is given by

$$\begin{aligned}\vec{x} &= (I - B\mathcal{E}_b)^{-1}\,\mathcal{E}_b\vec{y}\\ &= \left(\sum_{j=0}^{\infty} B^j\mathcal{E}_b^j\right)\mathcal{E}_b\vec{y}\end{aligned}$$

which in time series notation is

$$x_t = \sum_{j=0}^{\infty} B^j\mathbb{E}^b_{t,t+j}[y_{t+j+1}]$$

■

B.3 Proof of Lemma 2

Proof. η_t^U is related to forecast errors by

$$\eta_t^U = z_{UC}\left(C_t - \mathbb{E}^b_{t-1}[C_t]\right)$$

z_{UC} has more rows than columns, so the forecast error $C_t - \mathbb{E}^b_{t-1}[C_t]$ may be overdetermined.

Suppose a solution for η_t^U exists. Left multiply by M_{UC}:

$$M_{UC}\eta_t^U = 0$$

because $M_{UC}z_{UC} = 0$. In operator notation $\vec{\eta}^U = \Xi_{\eta,U}\vec{Y}^U$:

$$M_{UC}\Xi_{\eta,U}\vec{Y}^U = 0$$

which violates part (1) of Definition 5. No solution exists. ■

B.4 Proof of Proposition 1

Proof. There exists an eigenvector $\vec{u}_i$ such that

$$\left(\lambda_i^{\odot} - \mathcal{E}_b\right) \vec{u}_i = 0$$

and thus the vector $\vec{v} = (1 - \lambda_i^{\odot} L)\vec{u}_i$ satisfies

$$\left(\lambda_i^{\odot} - \mathcal{E}_b\right) (1 - \lambda_i^{\odot} L)^{-1} \vec{v} = 0$$

There is some matrix φ such that a single row in each block of $(\Lambda_S - \mathcal{E}_b)\,(I - \Lambda_S L)^{-1}\,(Q_S s_{\odot} \varphi)_i$ is nonzero, and that row is proportional to $\left(\lambda_i^{\odot} - \mathcal{E}_b\right)(1 - \lambda_i^{\odot} L)^{-1}$. Therefore

$$(\Lambda_S - \mathcal{E}_b)\,(I - \Lambda_S L)^{-1}\,(Q_S s_{\odot} \varphi)_i \vec{v} = 0$$

so the Sunspot Admissibility condition is satisfied. ■

B.5 Proof of Theorem 2

Proof. As in the proof of Theorem 1, χ_t^U and η_t^U are given by equations (11) and (12).

Suppose a solution exists. If $n_U \geq n_C$ then η_t can be inverted from η_t^U, so the solution is unique. If $n_U < n_C$, then η_t cannot be inverted from η_t^U; whether or not there exist multiple solutions depends on Sunspot Admissibility.

$\odot$ denotes the "sunspot" dimensions; let $\eta_t^{\odot}$ denote the corresponding rows of η_t^S, and let $\chi_t^{\odot}$ denote the corresponding rows of χ_t^S. The forecast errors can be found by

$$C_t - \mathbb{E}_{t-1}^b[C_t] = \begin{pmatrix} z_{\odot} \\ z_{UC} \end{pmatrix}^{-1} \begin{pmatrix} \eta_t^{\odot} \\ \eta_t^U \end{pmatrix}$$

and expected controls are given by

$$\mathbb{E}_t^b[C_{t+1}] = \begin{pmatrix} z_{\odot} \\ z_{UC} \end{pmatrix}^{-1} \begin{pmatrix} \chi_t^{\odot} \\ \chi_t^U \end{pmatrix}$$

$$\mathbb{E}_t^b\left[\begin{pmatrix} z_\odot \\ z_{UC} \end{pmatrix} C_{t+1}\right] = \begin{pmatrix} \chi_t^\odot \\ \chi_t^U \end{pmatrix}$$

$$\mathbb{E}_t^b\left[\begin{pmatrix} \eta_{t+1}^\odot \\ \eta_{t+1}^U \end{pmatrix} + \begin{pmatrix} \chi_t^\odot \\ \chi_t^U \end{pmatrix}\right] = \begin{pmatrix} \chi_t^\odot \\ \chi_t^U \end{pmatrix}$$

which in operator notation is

$$\mathcal{E}_b\left(\begin{pmatrix} \vec{\eta}^\odot \\ \vec{\eta}^U \end{pmatrix} + L\begin{pmatrix} \vec{\chi}^\odot \\ \vec{\chi}^U \end{pmatrix}\right) = \begin{pmatrix} \vec{\chi}^\odot \\ \vec{\chi}^U \end{pmatrix}$$

The unstable dimensions are determined by the unstable block, and satisfy this equation by construction. But, after rearranging, any solution for the sunspot dimensions must satisfy

$$\left(I - \mathcal{E}_b L\right)\vec{\chi}^\odot = \mathcal{E}_b\vec{\eta}^\odot \tag{27}$$

Denote the difference between two arbitrary solutions with hats, e.g. $\hat{\chi}^S \equiv \vec{\chi}_1^S - \vec{\chi}_2^S$, $\hat{\eta}^\odot \equiv \vec{\eta}_1^\odot - \vec{\eta}_2^\odot$, and so forth. These differences must satisfy equation (15) expressed in differences:

$$\hat{\chi}^S = T_{1,S}^{-1}T_{0,S}L\hat{\chi}^S + T_{1,S}^{-1}T_{0,S}\hat{\eta}^S$$

which is a simple expression because the unstable block's unique solution implies $\hat{\chi}^U = \hat{\eta}^U = 0$. This gives $\hat{\chi}^S$ in terms of $\hat{\eta}^S$:

$$\hat{\chi}^S = \left(I - T_{1,S}^{-1}T_{0,S}L\right)^{-1} T_{1,S}^{-1}T_{0,S}\hat{\eta}^S$$

Left multiply by the rectangular matrix $s_\odot'$ that isolates the sunspot rows:

$$\hat{\chi}^\odot = s_\odot'\left(I - T_{1,S}^{-1}T_{0,S}L\right)^{-1} T_{1,S}^{-1}T_{0,S}s_\odot\hat{\eta}^\odot$$

which also uses $\hat{\eta}^S = s_\odot\hat{\eta}^\odot$.

Eigendecompose by $T_{1,S}^{-1}T_{0,S} = Q_S^{-1}\Lambda_S Q_S$ to yield

$$\hat{\chi}^\odot = s_\odot'\left(I - Q_S^{-1}\Lambda_S Q_S L\right)^{-1} Q_S^{-1}\Lambda_S Q_S s_\odot\hat{\eta}^\odot$$

$$\hat{\chi}^\odot = s_\odot' Q_S^{-1}\left(I - \Lambda_S L\right)^{-1}\Lambda_S Q_S s_\odot\hat{\eta}^\odot$$

Plug in with equation (27) to find a single expression in $\hat{\eta}^{\odot}$ alone:

$$\mathcal{E}_b \hat{\eta}^{\odot} = s'_{\odot} Q_S^{-1} \left(I - \mathcal{E}_b L\right) \left(I - \Lambda_S L\right)^{-1} \Lambda_S Q_S s_{\odot} \hat{\eta}^{\odot}$$

Use $s'_{\odot} Q_S^{-1} Q_S s_{\odot} = I$ to collect terms:

$$0 = s'_{\odot} Q_S^{-1} \left(\left(I - \mathcal{E}_b L\right) \left(I - \Lambda_S L\right)^{-1} \Lambda_S - \mathcal{E}_b \right) Q_S s_{\odot} \hat{\eta}^{\odot}$$

Λ_S and $\left(I - \Lambda_S L\right)^{-1}$ commute:

$$0 = s'_{\odot} Q_S^{-1} \left(\left(I - \mathcal{E}_b L\right) \Lambda_S - \mathcal{E}_b \left(I - \Lambda_S L\right) \right) \left(I - \Lambda_S L\right)^{-1} Q_S s_{\odot} \hat{\eta}^{\odot}$$

$$0 = s'_{\odot} Q_S^{-1} \left(\Lambda_S - \mathcal{E}_b\right) \left(I - \Lambda_S L\right)^{-1} Q_S s_{\odot} \hat{\eta}^{\odot} \tag{28}$$

If and only if the Sunspot Admissibility condition hold, then there exists a vector $\vec{v}$ such that choosing any proportional vector $\hat{\eta}^{\odot} \propto \vec{v}$ satisfies equation (28). Thus multiple solutions exist. ■

B.6 Proof of Corollary 1

Proof. Sufficiency is given by Theorem 1. It remains to show that a solution cannot be unique if neither the Behavioral Blanchard-Kahn nor Sunspot Admissibility conditions hold. This occurs in three cases:

1. If $n_U < n_C$, then a unique solution does not exist by Theorem 2.
2. If $n_U = n_C$ and $n_S < n_K$, then there is no solution by Lemma 2.
3. If $n_U > n_C$, then it must be that $n_S < n_K$, so there is no solution by Lemma 2.

■

B.7 Proof of Proposition 2

Proof. The proof is cleanest in operator notation. Equation (24) becomes

$$\left(\eta + \frac{\alpha \gamma L}{1 - \delta L}\right) \hat{p} = (1 - \phi) \mathcal{E} \hat{p} + \phi \theta \hat{p} \tag{29}$$

Using the definition $\zeta \equiv \delta\eta - \alpha\gamma$:

$$(\eta - \zeta L)\,\hat{p} = (1 - \delta L)\,((1-\phi)\mathcal{E} + \phi\theta)\,\hat{p}$$

Rearrange:

$$0 = ((1-\phi)\mathcal{E} + (\phi\theta - \eta) - \delta(1-\phi)L\mathcal{E} + (\zeta - \delta\phi\theta)L)\,\hat{p} \tag{30}$$

Apply $\mathcal{E}$, using $\mathcal{E}L = I$:

$$0 = \big((1-\phi)\mathcal{E}^2 + (\phi\theta - \eta - \delta(1-\phi))\mathcal{E} + \zeta - \delta\phi\theta\big)\,\hat{p} \tag{31}$$

$$0 = \mathcal{E}^2\left(1 + \frac{\phi\theta - \eta - \delta(1-\phi)}{1-\phi}L + \frac{\zeta - \delta\phi\theta}{1-\phi}L^2\right)\hat{p}$$

without loss of generality, write $\hat{p} = \frac{\sum_{j=0}^{\infty} a_j L^j}{1 + \frac{\phi\theta-\eta-\delta(1-\phi)}{1-\phi}L + \frac{\zeta-\delta\phi\theta}{1-\phi}L^2}\upsilon$ for some white noise υ with $a_0 = 1$. This implies

$$0 = \mathcal{E}^2\left(1 + \frac{\phi\theta - \eta - \delta(1-\phi)}{1-\phi}L + \frac{\zeta - \delta\phi\theta}{1-\phi}L^2\right)\frac{\sum_{j=0}^{\infty} a_j L^j}{1 + \frac{\phi\theta-\eta-\delta(1-\phi)}{1-\phi}L + \frac{\zeta-\delta\phi\theta}{1-\phi}L^2}\upsilon$$

$$0 = \mathcal{E}^2\sum_{j=0}^{\infty} a_j L^j \upsilon_t = \mathcal{E}^2(1 + a_1 L)\upsilon$$

i.e. $a_j = 0$ for all $j > 1$. To solve for a_1, rewrite equation (30):

$$0 = ((1-\phi)\mathcal{E} + (\phi\theta - \eta - \delta(1-\phi)) + \delta(1-\phi)(1 - L\mathcal{E}) + (\zeta - \delta\phi\theta)L)\,\hat{p}$$

$$0 = \mathcal{E}\left(1 + \frac{\phi\theta - \eta - \delta(1-\phi)}{1-\phi}L + \frac{\zeta - \delta\phi\theta}{1-\phi}L^2\right)\hat{p} + \delta(1 - L\mathcal{E})\hat{p}$$

Plug in for $\hat{p}$:

$$0 = \mathcal{E}(1 + a_1 L)\upsilon + \delta(1 - L\mathcal{E})\frac{\sum_{j=0}^{\infty} a_j L^j}{1 + \frac{\phi\theta-\eta-\delta(1-\phi)}{1-\phi}L + \frac{\zeta-\delta\phi\theta}{1-\phi}L^2}\upsilon$$

and use the forecast $\mathcal{E}\upsilon = 0$ and the forecast error $(1 - L\mathcal{E})\frac{\sum_{j=0}^{\infty} a_j L^j}{1 + \frac{\phi\theta-\eta-\delta(1-\phi)}{1-\phi}L + \frac{\zeta-\delta\phi\theta}{1-\phi}L^2}\upsilon = \upsilon$ to yield

$$0 = a_1\upsilon + \delta\upsilon$$

$$\implies a_1 = -\delta$$

Any sunspot process must satisfy $\hat{p} = \frac{1-\delta L}{1+\frac{\phi\theta-\eta-\delta(1-\phi)}{1-\phi}L+\frac{\zeta-\delta\phi\theta}{1-\phi}L^2}\upsilon$; when is such a process stationary? When the representation $(1-r_2L)(1-\rho_2L) = 1+\frac{\phi\theta-\eta-\delta(1-\phi)}{1-\phi}L+\frac{\zeta-\delta\phi\theta}{1-\phi}L^2$ has roots ρ_1 and ρ_2 inside the unit disc. These roots are also the roots of the polynomial

$$z^2 + \frac{\phi\theta - \eta - \delta(1-\phi)}{1-\phi}z + \frac{\zeta - \delta\phi\theta}{1-\phi}$$

It is well known when quadratic roots are inside the unit disk: if and only if $|\frac{\zeta-\delta\phi\theta}{1-\phi}| < 1$ and $|\frac{\phi\theta-\eta-\delta(1-\phi)}{1-\phi}| < 1 + \frac{\zeta-\delta\phi\theta}{1-\phi}$. The assumption that $\phi \in (0,1)$ completes the proof. ■

B.8 Proof of Proposition 3

Proof. An expectation is strictly backwards-looking if its operator representation is a causal lag operator polynomial. These operators are Toeplitz and lower triangular (Adams, 2021). Per Property 3, it has no eigenvalues. Therefore the Sunspot Admissibility condition is not satisfied, and by Theorem 2 it cannot have multiple solutions. ■

B.9 Proof of Theorem 4

Proof. Apply the Schur decomposition to the rational expectations model in equation (4):

$$T_1 Z\mathbb{E}_t[X_{t+1}] = T_0 Z X_t + Q' B_Y Y_t$$

Take expectations of both sides

$$T_1 Z\mathbb{E}_{t-1}\left[\mathbb{E}_t[X_{t+1}]\right] = T_0 Z\mathbb{E}_{t-1}[X_t] + Q' B_Y \mathbb{E}_{t-1}[Y_t]$$

then isolate the unstable block using the definitions in equation (9)

$$T_{1,U}\mathbb{E}_t[\chi^U_{t+1}] = T_{0,U}\chi^U_t + \mathbb{E}_t[Y^U_{t+1}]$$

Rearrange and recur:

$$\chi^U_t = -\mathbb{E}_t\left[\sum_{j=0}^{\infty}\left(T_{0,U}^{-1}T_{1,U}\right)^j T_{0,U}^{-1}Y^U_{t+1}\right]$$

and replace the rational expectations operator to define the unstable block of the naive forecast:

$$\chi_{N,t}^{U} = -\mathbb{E}_t^b \left[\sum_{j=0}^{\infty} \left(T_{0,U}^{-1} T_{1,U} \right)^j T_{0,U}^{-1} Y_{t+1}^{U} \right] \tag{32}$$

The infinite sum converges because all eigenvalues of $T_{0,U}^{-1} T_{1,U}$ are inside the unit circle.

The unstable block of equation (25) is given by

$$T_{1,U} \chi_{N,t}^{U} = T_{0,U} \chi_{N,t-1}^{U} + T_{0,U} \eta_{N,t}^{U} + Y_t^{U}$$

which implies the unstable block of the naive forecast error is

$$\eta_{N,t}^{U} = T_{0,U}^{-1} T_{1,U} \chi_{N,t}^{U} - \chi_{N,t-1}^{U} - T_{0,U}^{-1} Y_t^{U} \tag{33}$$

where $\chi_{N,t}^{U}$ is given by equation (32).

As in the sophisticated case, naive agents forecast state variables without error, so the naive forecast error of the stable block is given by

$$\eta_{N,t}^{S} = z_{SC} z_{UC}^{-1} \eta_{N,t}^{U} \tag{34}$$

Finally, solutions for $\chi_{N,t}^{U}$, $\eta_{N,t}^{U}$, and $\eta_{N,t}^{S}$ allow the remaining unknown $\chi_{N,t}^{S}$ to be solved from the stable block of equation (25):

$$T_{1,S} \chi_{N,t}^{S} + T_{1,SU} \chi_{N,t}^{U} = T_{0,S} \chi_{N,t-1}^{S} + T_{0,SC} \chi_{N,t-1}^{U} + T_{0,S} \eta_{N,t}^{S} + T_{0,SC} \eta_{N,t}^{U} + Y_t^{S} \tag{35}$$

and rearrange for $\chi_{N,t}^{S}$:

$$\chi_{N,t}^{S} = T_{1,S}^{-1} T_{0,S} \chi_{N,t-1}^{S} - T_{1,S}^{-1} T_{1,SU} \chi_{N,t}^{U} + T_{1,S}^{-1} T_{0,SC} \chi_{N,t-1}^{U} + T_{1,S}^{-1} T_{0,S} \eta_{N,t}^{S} + T_{1,S}^{-1} T_{0,SC} \eta_{N,t}^{U} + T_{1,S}^{-1} Y_t^{S}$$

The matrix $T_{1,S}^{-1} T_{0,S}$ has all eigenvalues inside the unit circle, so $\chi_{N,t}^{S}$ can be written as

$$\chi_{N,t}^{S} = \sum_{j=0}^{\infty} \left(T_{1,S}^{-1} T_{0,S} \right)^j \left(-T_{1,SU} \chi_{N,t-j}^{U} + T_{0,SC} \chi_{N,t-1-j}^{U} + T_{0,S} \eta_{N,t-j}^{S} + T_{0,SC} \eta_{N,t-j}^{U} + Y_{t-j}^{S} \right) \tag{36}$$

Z is unitary, so the time series that uniquely solve the naive model (25) are recovered by

$$\chi_{N,t} = Z^* \begin{pmatrix} \chi_{N,t}^S \\ \chi_{N,t}^U \end{pmatrix}$$

$$X_{N,t} = \chi_{N,t} + \eta_{N,t} = \chi_{N,t} + Z^* \begin{pmatrix} \eta_{N,t}^S \\ \eta_{N,t}^U \end{pmatrix}$$

■

ONLINE APPENDIX

C Deriving the Behavioral New Keynesian Model

This section derives the behavioral New Keynesian equations of Section 5 from microfoundations.

C.1 Households

The representative household's problem is represented by the Bellman equation

$$V(A;Z) = \max_{C,N,B'} \frac{C^{1-\gamma}-1}{1-\gamma} - \chi\frac{N^{1+\eta}}{1+\eta} + \beta\mathbb{E}^b\left[V(A';Z')|Z\right]$$

$$\text{s.t.} \qquad AR + WN + D = C + QA'$$

The household's endogenous state variable is assets A which are purchased at a price Q set by the government and pay return R. The household's budget constraint is real: workers choose consumption C (the numeraire) and work hours N earning real wage W. Households also receive real dividends D from the firms, which they own. The vector Z includes exogenous state variables, profits, and prices, which atomistic households take as exogenous. $\mathbb{E}^b$ represents the household's behavioral expectation, and primes denote the next period's values.

The household's problem is solved by a labor supply equation

$$\chi N^{\eta} = WC^{-\gamma}$$

and an Euler equation

$$QC^{-\gamma} = \beta\mathbb{E}^b\left[\left(C'\right)^{-\gamma}R'\right]$$

The Euler equation can be derived as usual because $\mathbb{E}^b$ is assumed to be linear: the partial derivative operator passes through it, so that $\frac{\partial}{\partial B'}\mathbb{E}^b\left[V(B';Z')|Z\right] = \mathbb{E}^b\left[\frac{\partial}{\partial B'}V(B';Z')|Z\right]$

The asset structure is unusual for a New Keyensian model. Q captures the component of returns that is known in advance, while R captures the component that is stochastic. Under rational expectations, both components can be included in R without affecting behavior; this is not true when expectations are behavioral. Different assumptions about Q imply different equilibria. In this case, I assume:

$$Q = \frac{1}{1+i} \qquad R = \frac{1}{\Pi}$$

where i is the nominal interest rate set by monetary policy and $\Pi = \frac{P'}{P}$ is inflation in the price of consumption. These assumptions imply that the Euler equation becomes:

$$C^{-\gamma} = (1+i)\beta\mathbb{E}^b\left[(C')^{-\gamma}\frac{1}{\Pi'}\right] \tag{37}$$

C.2 Firms

The final good Y is produced by a competitive retail sector, which aggregates firm-specific intermediate goods indexed by $j \in [0,1]$:

$$Y = \left(\int_{j=0}^{1}(Y_j)^{\frac{\epsilon-1}{\epsilon}}dj\right)^{\frac{\epsilon}{\epsilon-1}}$$

which implies the standard demand function

$$\left(\frac{Y}{Y_j}\right)^{\frac{1}{\epsilon}} = \frac{P_j'}{P}$$

There is a unit measure of monopolistic firms producing intermediate goods. Firm j's real dividends are given by

$$D_j = \frac{P_j'}{P}Y_j - WN_j$$

where P_j' is the price of their output Y_j. They produce by hiring N_j workers:

$$Y_j = \zeta N_j$$

where ζ is total factor productivity.

Each firm maximizes the present value of real dividends, which it discounts in the same way as the household. Firms face a Rotemberg (1982) adjustment cost when changing prices. When selecting a new price P_j', the firm pays real cost $\psi/2(\log P_j' - \log P_j)^2$ in units of the numeraire, where P_j is the firm's price in the previous period.

The firm's value function is

$$V(P_j, Z) = \max_{P'_j, N} \frac{P'_j}{P}\zeta N_j - WN_j - \psi/2(\log P'_j - \log P_j)^2 + C^\gamma \beta \mathbb{E}^b \left[(C')^{-\gamma} V(P'_j, Z')|Z \right]$$

subject to the retail demand function $\frac{P_j}{P} = \left(\frac{Y}{AN_j}\right)^{\frac{1}{\epsilon}}$. Written in terms of a single choice variable, the value function is

$$V(P_j, Z) = \max_{P'_j} \left(\frac{P'_j}{P}\right)^{1-\epsilon} Y - \frac{W}{\zeta}\left(\frac{P'_j}{P}\right)^{-\epsilon} Y - \psi/2(\log P'_j - \log P_j)^2 + C^\gamma \beta \mathbb{E}^b \left[(C')^{-\gamma} V(P'_j, Z')|Z \right]$$

The Euler equation is where $\Pi_j \equiv \frac{P'_j}{P_j}$

$$0 = (1-\epsilon)\left(\frac{P'_j}{P}\right)^{1-\epsilon} Y + \epsilon \frac{W}{\zeta}\left(\frac{P_j}{P}\right)^{-\epsilon} Y - \psi \log \Pi_j + P_j C^\gamma \beta \mathbb{E}^b \left[\frac{1}{P'_j} (C')^{-\gamma} \psi \log \Pi'_j | Z \right]$$

C.3 Aggregation and Market Clearing

Impose symmetry:

$$0 = (1-\epsilon)Y + \epsilon \frac{W}{\zeta} Y - \psi \log \Pi + PC^\gamma \beta \mathbb{E}^b \left[\frac{1}{P'} (C')^{-\gamma} \psi \log \Pi' | Z \right]$$

The real wage is determined from the labor supply equation as $W = \chi N^\eta C^\gamma$. This expression can be written in terms of output, using $N = Y/\zeta$ and $C = Y$ to get $W = \chi \zeta^{-\eta} Y^{\eta+\gamma}$, implying

$$0 = (1-\epsilon)Y + \epsilon \chi \zeta^{-\eta-1} Y^{\eta+\gamma+1} - \psi \log \Pi + PC^\gamma \beta \mathbb{E}^b \left[\frac{1}{P'} (C')^{-\gamma} \psi \log \Pi' | Z \right]$$

Log-linearize around the deterministic steady state, where lowercase variables denote log deviations, and over-bars denote steady state values:

$$0 = \left((1-\epsilon)\bar{Y} + \epsilon \chi \zeta^{-\eta-1} \bar{Y}^{\eta+\gamma+1}(\eta+\gamma+1)\right) y - \psi \pi + \psi \beta \mathbb{E}^b \left[\pi' | Z\right]$$

Rearranging gives the New Keynesian Phillips Curve with $\kappa \equiv \frac{(1-\epsilon)\bar{Y} + \epsilon \chi \zeta^{-\eta-1} \bar{Y}^{\eta+\gamma+1}(\eta+\gamma+1)}{\psi}$:

$$\pi = \kappa y + \beta \mathbb{E}^b \left[\pi' | Z\right]$$

or in the time series notation

$$\pi_t = \kappa y_t + \beta \mathbb{E}_t^b [\pi_{t+1}]$$

Similarly, log-linearize the Euler equation (37) and impose market clearing so that $C = Y$:

$$i_t = \mathbb{E}^b [\gamma y_{t+1} + \pi_{t+1}] - \gamma y_t$$

This gives the two microfounded equations of the behavioral New Keynesian model in Section 5; the assumed Taylor rule completes the model.

D Deriving the Spectra of Various Expectations

This appendix motivates the operator representations of the expectations appearing in Table 2, and describes their spectra.

A series-agnostic expectations operator $\mathcal{E}_b$ operating on the vector of n time series X_t has the same spectral radius $r(\mathcal{E}_b)$ for all $n \geq 1$. Thus in the following sections I describe the operators with scalar blocks, but the operator blocks expand to conform to the dimension of the time series on which they operate, without affecting any features of their spectra.

D.1 Rational Expectations

Let $\mathcal{E}$ without subscript denote the one-period-ahead rational expectation operator. The rational expectation $\mathbb{E}_t[X_{t+1}]$ of the time series $X_t = \sum_{j=0}^{\infty} X_j L^j \omega_t$ is

$$\mathbb{E}_t[X_{t+1}] = \mathbb{E}_t[\sum_{j=0}^{\infty} X_j L^j \omega_{t+1}]$$

$$= \sum_{j=0}^{\infty} X_{j+1} L^j \omega_t$$

The block vector representation of the one-period-ahead rational expectation is

$$\mathcal{E}\vec{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \end{pmatrix}$$

Therefore the operator $\mathcal{E}$ is given by

$$\mathcal{E} = \begin{pmatrix} 0 & 1 & 0 & 0 & ... \\ 0 & 0 & 1 & 0 & ... \\ 0 & 0 & 0 & 1 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = L'$$

$\mathcal{E}$ is Toeplitz with largest entry 1, so the spectral radius is

$$r(\mathcal{E}) = 1$$

which is why the traditional Blanchard-Kahn condition is a special case of the Behavioral Blanchard-Kahn condition.

The Sunspot Admissibility condition is always satisfied for rational expectations. Any stable eigenvalue λ is also an eigenvalue of $\mathcal{E}$, with eigenvector corresponding to the AR(1) process $y_t = \lambda y_{t-1} + \omega_t$:

$$\mathbb{E}_t[y_{t+1}] = \lambda y_t$$

To satisfy Sunspot Admissibility, there must exist a time series represented in vector form $\vec{v}$ such that

$$s'_{\odot} Q_S^{-1} \left(\Lambda_S - \mathcal{E}_b\right) \left(I - \Lambda_S L\right)^{-1} Q_S s_{\odot} \vec{v} = 0$$

If $\vec{v}$ is a white noise process, then $Q_S s_{\odot} \vec{v}$ is also white noise, $\left(I - \Lambda_S L\right)^{-1} Q_S s_{\odot} \vec{v}$ is AR(1), and

$$\mathcal{E} \left(I - \Lambda_S L\right)^{-1} Q_S s_{\odot} \vec{v} = \Lambda_S \left(I - \Lambda_S L\right)^{-1} Q_S s_{\odot} \vec{v}$$

satisfying the property.

D.2 Sub-rational Expectations

This section describes several types of "sub-rational" expectations. These expectations feature behavioral biases and predictable forecast errors. Yet, they preserve several features of rational expectations: they are forward-looking, and depend on equilibrium behavior in the modeled economy.

D.2.1 Mis-extrapolation

"Mis-extrapolation" modifies the rational expectation by the scalar $\theta > 0$:

$$\mathbb{E}_t^{ME,\theta} x_{t+1} = \theta \mathbb{E}_t x_{t+1}$$

This form of expectations is known by different names in different cases. In Gabaix (2020), agents form expectations by "cognitive discounting", which is carefully microfounded, but manifests in reduced form as forecasting with $0 < \theta < 1$. Angeletos, Huo, and Sastry (2021) suggest that to fit facts about the term structure of expectations, it may be worth considering "overextrapolation" where $\theta > 1$.

The operator form of mis-extrapolation is

$$\mathcal{E}_{ME,\theta} = \theta \mathcal{E} = \begin{pmatrix} 0 & \theta & 0 & 0 & \ldots \\ 0 & 0 & \theta & 0 & \ldots \\ 0 & 0 & 0 & \theta & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with spectral radius

$$r(\mathcal{E}_{ME,\theta}) = \theta r(\mathcal{E}) = \theta$$

where the first step follows from the absolute homogeneity of spectral radii.

An eigenvalue λ of $\mathcal{E}_{ME,\theta}$ satisfies

$$\lambda y_t = \mathbb{E}_t^{ME,\theta}[y_{t+1}]$$

$$= \theta \mathbb{E}_t[y_{t+1}]$$

which is solved by eigenseries

$$y_t = \frac{\lambda}{\theta} y_{t-1} + \omega_t$$

Such a series may not satisfy Sunspot Admissibility in general, but will for simply stable models where $|\frac{\lambda}{\theta}| < 1$ so that y_t is stationary.

D.2.2 Diagnostic Expectations

Diagnostic expectations (Bordalo, Gennaioli, and Shleifer, 2018) are growing in popularity, have many appealing properties, and describe a variety of empirical patterns.[16] This section demonstrates another useful property: the spectral radius is 1. This implies that models with diagnostic expectations have unique equilibria if they would also have unique equilibria under rational expectations.

The diagnostic expectation is given by

$$\mathbb{E}_t^{D,\theta,\phi} x_{t+1} = \mathbb{E}_t x_{t+1} + \theta(\mathbb{E}_t x_{t+1} - \mathbb{E}_{t-\phi} x_{t+1})$$

where θ controls the degree of overreaction to recent news, while the integer ϕ controls the number of recent periods to which agents overreact. The operator representation is

$$\mathcal{E}_{D,\theta,\phi} = \mathcal{E} + \theta(\mathcal{E} - L^{\phi}\mathcal{E}^{\phi+1})$$

$$= \begin{pmatrix} 0 & 1+\theta & 0 & 0 & 0 & ... \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & ... & 0 & 1+\theta & 0 & \ddots \\ 0 & ... & 0 & 0 & 1+\theta & \ddots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & ... \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & ... & 0 & 0 & 0 & \ddots \\ 0 & ... & 0 & 0 & \theta & \ddots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1+\theta & 0 & 0 & 0 & ... \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & ... & 0 & 1+\theta & 0 & \ddots \\ 0 & ... & 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & \vartheta_1^{D,\theta,\phi} & 0 & 0 & 0 & ... \\ 0 & 0 & \vartheta_2^{D,\theta,\phi} & 0 & 0 & \ddots \\ 0 & 0 & 0 & \vartheta_3^{D,\theta,\phi} & 0 & \ddots \\ 0 & 0 & 0 & 0 & \vartheta_4^{D,\theta,\phi} & \ddots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots \end{pmatrix}$$

[16] See for example Bordalo, Gennaioli, Porta, and Shleifer (2019), Bordalo, Gennaioli, Ma, and Shleifer (2020), or Chodorow-Reich, Guren, and McQuade (2021).

where the diagonal series is given by

$$\vartheta_i^{D,\theta,\phi} = \begin{cases} 1+\theta & i \leq \phi \\ 1 & i > \phi \end{cases}$$

The diagnostic expectation operator is not Toeplitz, so its spectral radius is not as straightforward as many of the other expectation operators. In these cases, it is sometimes possible to derive the spectral radius analytically by characterizing powers of the operator. The iterated expectation $\mathcal{E}_{D,\theta,\phi}^k$ is zero except for the kth diagonal. The largest entry is $(1+\theta)^k$ when $k \leq \phi$, and $(1+\theta)^\phi$ otherwise. The norm of a diagonal operator is the largest magnitude entry, implying

$$||\mathcal{E}_{D,\theta,\phi}^k|| = \begin{cases} |(1+\theta)^k| & k \leq \phi \\ |(1+\theta)^\phi| & k > \phi \end{cases}$$

By Gelfand's formula (26) the spectral radius is

$$r(\mathcal{E}_{D,\theta,\phi}) = \lim_{k\to\infty} ||\mathcal{E}_{D,\theta,\phi}^k||^{\frac{1}{k}} = \lim_{k\to\infty} |(1+\theta)^\phi|^{\frac{1}{k}} = 1$$

Under Diagnostic Expectations, Sunspot Admissibility may not be satisfied in general,[17] but will be for simply stable models. An eigenvalue λ of $\mathcal{E}_{D,\theta,\phi}$ satisfies

$$\lambda y_t = \mathbb{E}_t^{D,\theta,\phi}[y_{t+1}]$$

$$= \mathbb{E}_t[y_{t+1}] + \theta\left(\mathbb{E}_t[y_{t+1}] - \mathbb{E}_{t-\phi}[y_{t+1}]\right)$$

To find the eigenseries, consider the moving average representation $y_t = \sum_{j=0}^{\infty} a_j L^j \omega_t$. The coefficients are related by

$$\lambda a_{j-1} = \begin{cases} (1+\theta)a_j & j \leq \phi \\ a_j & j > \phi \end{cases}$$

[17]However, I suspect that Sunspot Admissibility *is* always satisfied for Diagnostic Expectations: I have found no counterexamples, but cannot prove the conjecture.

Normalizing $a_0 = 1$ implies

$$a_j = \begin{cases} \left(\frac{\lambda}{1+\theta}\right)^j & j < \phi \\ \left(\frac{\lambda}{1+\theta}\right)^\phi \lambda^{j-\phi} & j \geq \phi \end{cases}$$

so the two components of the moving average representation are

$$\sum_{j=0}^{\phi-1} a_j L^j = \frac{1}{1 - \frac{\lambda}{1+\theta}L} - \left(\frac{\lambda}{1+\theta}L\right)^\phi \frac{1}{1 - \frac{\lambda}{1+\theta}L}$$

$$\sum_{j=\phi}^{\infty} a_j L^j = \left(\frac{\lambda}{1+\theta}L\right)^\phi \frac{1}{1-\lambda L}$$

and the eigenseries is

$$y_t = \left(\frac{1 - \left(\frac{\lambda}{1+\theta}L\right)^\phi}{1 - \frac{\lambda}{1+\theta}L} + \frac{\left(\frac{\lambda}{1+\theta}L\right)^\phi}{1-\lambda L}\right)\omega_t$$

This is a stationary ARMA(2,$\phi + 1$) time series for any stable λ, except in the case of $\phi = 1$ where it reduces to an ARMA(1,1):

$$[\phi = 1]: \qquad y_t = \frac{1 - \frac{\lambda\theta}{1+\theta}L}{1-\lambda L}\omega_t$$

D.2.3 Delayed Observation

Many macroeconomic time series are only measured and publicly released with a delay. "Delayed Observation" expectations reflect this constraint. Forecasters cannot use the current value of a time series; they forecast rationally, but only using data realized in the past. This structure has a representation as a behavioral expectation, even though agents are fully rational.

The "DO"-type expectation is given by

$$\mathbb{E}_t^{DO} x_{t+1} = \mathbb{E}_{t-1} x_{t+1}$$

which in operators is

$$\mathcal{E}_{DO} = L\mathcal{E}^2$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & ... \\ 0 & 0 & 1 & 0 & ... \\ 0 & 0 & 0 & 1 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Iterating $\mathcal{E}_{DO}$ multiple times gives

$$\mathcal{E}_{DO}^k = \left(L\mathcal{E}^2\right)^k = L\mathcal{E}^{k+1}$$

because $\mathcal{E}L = I$. Additionally, the norm is $||L\mathcal{E}^{k+1}|| = 1$, because the norm of a diagonal operator is the supremum of the magnitude of its entries. This allows the spectral radius is found by Gelfand's formula (26):

$$r(\mathcal{E}_{DO}) = \lim_{k\to\infty} ||\mathcal{E}_{DO}^k||^{\frac{1}{k}} = \lim_{k\to\infty} |1|^{\frac{1}{k}} = 1$$

The Sunspot Admissibility condition is always satisfied for DO-type expectations. Any stable eigenvalue λ is also an eigenvalue of $\mathcal{E}_{DO}$, which satisfies

$$\lambda y_t = \mathbb{E}_t^{DO}[y_{t+1}]$$

$$= \mathbb{E}_{t-1}[y_{t+1}]$$

which is solved by eigenseries

$$y_t = \lambda y_{t-1} + \omega_{t-1}$$

Just as rational expectations sunspot equilibria are created with any white noise process, delayed observation sunspot equilibria can be created with any *lagged* white noise process.

D.2.4 Sticky Information

Mankiw and Reis (2002) introduce "Sticky Information" in which agents only update their forecasts with some probability $\theta \in (0, 1)$. The Sticky Information expectations are given by

$$\mathbb{E}_t^{SI,\theta} x_{t+1} = \theta \sum_{j=0}^{\infty} (1-\theta)^j \mathbb{E}_{t-j} x_{t+1}$$

In Mankiw and Reis' original framework, no individual forms expectations this way. Rather, these expectations govern the average expectations in the economy.[18] Moreover, in their original application these expectations are not used to forecast a process x_{t+1} but rather to nowcast an optimal price-setting decision. Still, other applications use Sticky Information to replace rational expectations in more general settings.

When forecasting x_{t+1}, the operator form is

$$\mathcal{E}_{SI,\theta} = \theta \sum_{j=0}^{\infty} (1-\theta)^j L^j \mathcal{E}^{j+1}$$

$$= \theta \begin{pmatrix} 0 & 1 & 0 & 0 & ... \\ 0 & 0 & 1 & 0 & ... \\ 0 & 0 & 0 & 1 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \theta \begin{pmatrix} 0 & 0 & 0 & 0 & ... \\ 0 & 0 & 1-\theta & 0 & ... \\ 0 & 0 & 0 & 1-\theta & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \theta \begin{pmatrix} 0 & 0 & 0 & 0 & ... \\ 0 & 0 & 0 & 0 & ... \\ 0 & 0 & 0 & (1-\theta)^2 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + ...$$

$$= \begin{pmatrix} 0 & \vartheta_1^{SI,\theta} & 0 & 0 & ... \\ 0 & 0 & \vartheta_2^{SI,\theta} & 0 & \ddots \\ 0 & 0 & 0 & \vartheta_3^{SI,\theta} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

where the diagonal series is given by

$$\vartheta_i^{SI,\theta} = \theta \sum_{k=0}^{i-1} (1-\theta)^k$$

By assumption $\theta \in (0,1)$ so this sequence satisfies $\vartheta_j^{SI,\theta} \in (0,1)$ with limit

$$\lim_{j\to\infty} \theta \sum_{k=0}^{j} (1-\theta)^k = 1$$

As with diagnostic expectations, this operator is not Toeplitz, but the spectral radius can be found using Gelfand's formula (26). The iterated expectation $\mathcal{E}_{SI,\theta}^b$ has

[18] Closely related is the "6D bias" (Gabaix and Laibson, 2001) where agents deterministically update every D periods, instead of stochastically. In this case, the operator representation is $\mathcal{E}^{6D} = \sum_{h=0}^{D-1} \frac{1}{D} L^h \mathcal{E}^{h+1}$. As with the sticky information formulation, the $6D$ bias operator is sub-rational, has a unit spectral radius, and SSA is satisfied for all simply stable models.

only one non-zero diagonal D_k with entries given by

$$D_{k,i} = \prod_{j=0}^{k-1} \vartheta_{i+j}^{SI,\theta}$$

This operator is diagonal, so the norm is given by

$$||\mathcal{E}_{SI,\theta}^k|| = \sup_i |\prod_{j=0}^{k-1} \vartheta_{i+j}^{SI,\theta}|$$

$$= \lim_{i\to\infty} |\prod_{j=0}^{k-1} \vartheta_{i+j}^{SI,\theta}| = 1$$

Thus by Gelfand's formula the spectral radius is

$$r(\mathcal{E}_{SI,\theta}) = \lim_{k\to\infty} ||\mathcal{E}_{SI,\theta}^k||^{\frac{1}{k}} = 1$$

An eigenvalue λ of $\mathcal{E}_{SI,\theta}$ satisfies

$$\lambda y_t = \mathbb{E}_t^{ME,\theta}[y_{t+1}]$$

$$= \theta \left(\mathbb{E}_t[y_{t+1}] + (1-\theta)\mathbb{E}_{t-1}[y_{t+1}] + (1-\theta)^2\mathbb{E}_{t-2}[y_{t+1}] + ...\right)$$

The eigenseries for these expectations is complicated. Consider the moving average representation $y_t = \sum_{j=0}^{\infty} a_j L^j \omega_t$. The operator representation implies that the eigenseries associated with eigenvalue λ must satisfy the recursive relationship

$$\lambda a_{j-1} = \vartheta_j^{SI,\theta} a_j$$

so any particular coefficient is given by

$$a_j = \frac{\lambda^j}{\prod_{k=1}^j \vartheta_k^{SI,\theta}} a_0$$

Therefore a stationary eigenseries exists if $\lim_{j\to\infty} \frac{\lambda^j}{\prod_{k=1}^j \vartheta_k^{SI,\theta}} = 0$, which is the case if $|\lambda| < \lim_{j\to\infty} \vartheta_j^{SI,\theta} = 1$, i.e. for any stable λ.

D.3 Heuristic Expectations

Not all forms of expectations are forward-looking. Before the rational expectations revolution, modeled expectations were often backwards-looking heuristics. These types of expectations depend on current and past realizations with an assumed structure.

D.3.1 Adaptive Expectations

A classic heuristic is the Adaptive Expectations of Cagan (1956) and Friedman (1957).

The Adaptive Expectation of a time series is given recursively by

$$\mathbb{E}_t^{AE,\theta} x_{t+1} = \theta x_t + (1-\theta)\mathbb{E}_{t-1}^{AE,\theta} x_t$$

which in operator notation is

$$\begin{aligned}
\mathcal{E}_{AE,\theta} &= \theta + (1-\theta)L\mathcal{E}_{AE,\theta} \\
&= \frac{\theta}{1-(1-\theta)L} \\
&= \theta \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1-\theta & 1 & 0 & 0 & \dots \\ (1-\theta)^2 & 1-\theta & 1 & 0 & \dots \\ (1-\theta)^3 & (1-\theta)^2 & 1-\theta & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\end{aligned}$$

This operator is Toeplitz, so its spectral radius is equal to the absolute value of the largest entry:

$$r(\mathcal{E}_{AE,\theta}) = \theta$$

The Sunspot Admissibility condition is never satisfied for Adaptive Expectations: $\mathcal{E}_{AE,\theta}$ is Toeplitz and lower triangular, so by Property 3, it has no eigenvalues.

D.3.2 Naive Heuristic

Brock and Hommes (1997) consider a “naive” heuristic in which agents forecast using current or past realizations:[19]

$$\mathbb{E}^{NH,\theta}\left[x_{t+1}\right] = x_{t-\theta}$$

for $\theta \geq 0$.

The corresponding expectation operator is

$$\mathcal{E}_{NH,\theta} = L^{\theta}$$

with spectral radius

$$r(\mathcal{E}_{NH,\theta}) = r(L^{\theta}) = 1$$

The Sunspot Admissibility condition is never satisfied for the Naive Heuristic: L^{θ} is Toeplitz and lower triangular, so by Property 3, it has no eigenvalues.

D.3.3 Behavioral Learning

Tuinstra (2003) and Hommes and Zhu (2014) consider "Behavioral Learning," where agents forecast using an incorrectly specified AR(1) model, which may not match the rational forecast if the true model is not AR(1). However, this mapping is not a linear operator, so the uniqueness theorems will not apply.

Still, it may be valuable to understand the properties of these expectations for an arbitrary AR(1) model, rather than the endogenous optimally-estimated model. Expectations of this form are given by

$$\mathbb{E}^{BL,\theta}\left[x_{t+1}\right] = \theta x_t$$

where θ is the first autocorrelation of the x_t time series. The expectation operator for time series x is

$$\mathcal{E}_{BL,\theta} = \theta I$$

with spectral radius

$$r(\mathcal{E}_{BL,\theta}) = |\theta| < 1$$

which is necessarily less than one.

[19] In their original work and many following papers, Brock and Hommes name this heuristic "naive expectations", which I relabel to avoid confusion with the naivete discussed in Section 6.

The Sunspot Admissibility condition is never satisfied for Behavioral Learning: $\mathcal{E}_{BL,\theta}$ is Toeplitz and lower triangular, so by Property 3, it has no eigenvalues.

D.3.4 Natural Expectations

Fuster, Laibson, and Mendel (2010) consider "Natural Expectations", which is a linear combination of rational expectations and some "intuitive" form of forecasting, with relative weights controlled by the parameter ϕ. In their original application, Fuster et al assume that the intuitive model is a parsimonious model, estimated on the equilibrium time series. For example, agents might estimate an AR(1), while the true process is a higher order ARMA. As with Behavioral Learning, this mapping is not a linear operator, so the uniqueness theorems will not apply, but it may be valuable to understand the properties of these expectations for any given AR(1) intuitive model, rather than the endogenous optimal model.

I approximate the Natural Expectations structure by assuming that the intuitive forecasting is given by some arbitrary heuristic AR(1), instead of being determined in equilibrium. These expectations are

$$\mathbb{E}^{NE,\theta,\phi}x_{t+1} = \phi\theta x_t + (1-\phi)\mathbb{E}_t x_{t+1}$$

where $\phi \in (0,1)$ denotes the relative weight placed on the intuitive forecast, and θ is the autocorrelation of the heuristic AR(1). Were $\phi = 1$, the expectations would simplify to Behavioral Learning, but with $\phi \neq 1$ Natural Expectations have more interesting spectra, e.g. eigenvalues exist.

The operator form is

$$\mathcal{E}_{NE,\theta,\phi} = \phi\theta + (1-\phi)\mathcal{E}$$

$$= \begin{pmatrix} \phi\theta & 1-\phi & 0 & 0 & ... \\ 0 & \phi\theta & 1-\phi & 0 & ... \\ 0 & 0 & \phi\theta & 1-\phi & ... \\ 0 & 0 & 0 & \phi\theta & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This operator is Toeplitz, and the largest magnitude entry is either $\phi\theta$ or $1-\phi$.

Accordingly, the spectral radius is

$$r(\mathcal{E}_{NE,\theta,\phi}) = \max(|\phi\theta|, 1-\phi)$$

An eigenvalue λ of $\mathcal{E}_{NE,\theta,\phi}$ satisfies

$$\lambda y_t = \mathbb{E}_t^{NE,\theta,\phi}[y_{t+1}]$$

$$= \phi\theta y_t + (1-\phi)\mathbb{E}_t[y_{t+1}]$$

which is solved by eigenseries

$$y_t = \frac{\lambda - \phi\theta}{1-\phi} y_{t-1} + \omega_t$$

Such a series may not satisfy Sunspot Admissibility in general, but will for simply stable models where $|\frac{\lambda-\phi\theta}{1-\phi}| < 1$ so that y_t is stationary.

D.3.5 Trend Following

Hommes, Massaro, and Weber (2019) consider "Trend Following" expectations, defined as

$$\mathbb{E}_t^{TF,\theta} x_{t+1} = x_{t-1} + \theta(x_{t-1} - x_{t-2})$$

where $\theta > 0$ may be larger or small than 1, which they label as strong or weak trend following respectively.

The operator form is

$$\mathcal{E}_{TF,\theta} = (1+\theta)L - \theta L^2$$

$$= \theta \begin{pmatrix} 0 & 0 & 0 & 0 & ... \\ 1+\theta & 0 & 0 & 0 & ... \\ -\theta & 1+\theta & 0 & 0 & ... \\ 0 & -\theta & 1+\theta & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This operator is Toeplitz, so its spectral radius is the largest absolute entry:

$$r(\mathcal{E}_{TF,\theta}) = 1 + \theta$$

The Sunspot Admissibility condition is never satisfied for TF-type expectations: $\mathcal{E}_{TF,\theta}$ is Toeplitz and lower triangular, so by Property 3, it has no eigenvalues.

D.3.6 Anchoring and Adjustment

Hommes, Massaro, and Weber (2019) consider another heuristic "Anchoring and Adjustment" which includes a term for the observed long-run average, motivated by Tversky and Kahneman (1974). Over time, the average goes to zero, leaving the limiting form of these expectations:

$$\mathbb{E}_t^{AA,\theta} x_{t+1} = \theta x_{t-1} + (x_{t-1} - x_{t-2})$$

$$\mathcal{E}_{AA,\theta} = (1+\theta)L - L^2$$

$$= \theta \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ 1+\theta & 0 & 0 & 0 & \ldots \\ -1 & 1+\theta & 0 & 0 & \ldots \\ 0 & -1 & 1+\theta & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Again, this operator is Toeplitz, so its spectral radius is the largest absolute entry:

$$r(\mathcal{E}_{AA,\theta}) = 1 + \theta$$

The Sunspot Admissibility condition is never satisfied for AA-type expectations: $\mathcal{E}_{AA,\theta}$ is Toeplitz and lower triangular, so by Property 3, it has no eigenvalues.

D.3.7 Heterogeneous Expectations with a Heuristic

Models with heterogeneous expectation formation can be expressed as a series-agnostic operator if agents' expectations can be aggregated when writing the dynamic model. This section considers such an example from Branch and McGough (2004).

A fraction $1 - \phi$ of agents have rational expectations. The remaining fraction ϕ form expectations with a heuristic: $\tilde{\mathbb{E}}[x_{t+1}] = \theta x_{t-1}$. The average in the economy defines the HEH-type expectations:

$$\mathbb{E}^{HEH,\theta,\phi}[x_{t+1}] = (1-\phi)\mathbb{E}[x_{t+1}] + \phi\theta x_{t-1}$$

The operator form is

$$\mathcal{E}_{HEH,\theta,\phi} = (1-\phi)\mathcal{E} + \phi\theta L$$

$$= \begin{pmatrix} 0 & 1-\phi & 0 & 0 & ... \\ \phi\theta & 0 & 1-\phi & 0 & ... \\ 0 & \phi\theta & 0 & 1-\phi & ... \\ 0 & 0 & \phi\theta & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This operator is Toeplitz, and the largest magnitude entry is either $1-\phi$ or $\phi\theta$. Accordingly, the spectral radius is

$$r(\mathcal{E}_{HEH,\theta,\phi}) = \max(1-\phi, |\phi\theta|)$$

An eigenvalue λ of $\mathcal{E}_{HEH,\theta,\phi}$ satisfies

$$\lambda y_t = \mathbb{E}_t^{HEH,\theta,\phi}[y_{t+1}]$$

$$= \phi\theta y_{t-1} + (1-\phi)\mathbb{E}_t[y_{t+1}]$$

which implies an AR(2) eigenseries:

$$y_{t+1} = \frac{\lambda}{1-\phi} y_t - \frac{\phi}{1-\phi}\theta y_{t-1} + \omega_t$$

This time series is stationary if the roots of the polynomial $z^2 - \frac{\lambda}{1-\phi}z + \frac{\phi}{1-\phi}\theta$ are inside the unit disc.

Therefore, such a series may not satisfy Sunspot Admissibility in general, but will for simply stable models where $|\theta| < \frac{1-\phi}{\phi}$ and $|\lambda| < 1-\phi+(\phi\theta)$ so that the eigenseries y_t is stationary.

D.4 Incomplete Information

Incomplete information is unlike the behavioral expectations considered thus far. Behavioral expectations are not rational, and the forecasts are spanned by the entire set of fundamental shocks ω_t. However, when information is incomplete, expectations may be rational, but are spanned by a different set of shocks: either noise shocks are added, or some linear combination of shocks is removed.

Still, in some simple cases, incomplete information can be represented as a behavioral expectation by considering the component that is spanned by the fundamental shocks alone. Many rich information structures may not be represented this way, such as when agents information set is endogenous. But this section explores some tractable cases where the incomplete information forecasts have a representation as behavioral forecasts.

D.4.1 Noisy Signals

Under this structure, agents form expectations about future variables using noisy signals of the fundamental shocks. The friction only affects expectation formation: the shocks can affect non-expectational variables and equations as usual, even if this introduces internal inconsistency in agents' information sets. The model is written as if all agents receive the same noisy signal. If instead agents receive private signals, then the expectations I derive represent the average agent.

I assume that when forming expectations, agents do not use the true values of the shocks ω_t. Instead, they observe a noisy signal s_t given by

$$s_t = \omega_t + \nu_t \tag{38}$$

where the noise term is orthogonal to the shocks and i.i.d. distributed $\nu_t \sim N(0, \theta)$. This structure generalizes when there is more than one shock so long as σ_ν^2 is the same for all shocks. Otherwise, expectations no longer satisfy Property 1.

Agents forecast by

$$\mathbb{E}_t^{NS}[X_{t+h}] = \mathbb{E}_t[X_{t+h}|\{s_{t-j}\}_{j=0}^{\infty}]$$

$$= \mathbb{E}_t[\sum_{j=0}^{\infty} X_{j+h}\omega_{t-j}|\{s_{t-j}\}_{j=0}^{\infty}] = \sum_{j=0}^{\infty} X_{j+h}\mathbb{E}_t[\omega_{t-j}|s_{t-j}]$$

$$= \sum_{j=0}^{\infty} X_{j+h}\frac{1}{1+\theta}s_{t-j}$$

To express the expectations with noisy signals as a behavioral expectation, consider only the effects of the fundamental shocks ω_t. This representation is:

$$\mathbb{E}[\mathbb{E}_t^{NS}[X_{t+h}]|\{\omega_{t-j}\}_{j=0}^{\infty}] = \mathbb{E}[\mathbb{E}[X_{t+h}|\{s_{t-j}\}_{j=0}^{\infty}]|\{\omega_{t-j}\}_{j=0}^{\infty}]$$

$$= E\left[\sum_{j=0}^{\infty} X_{j+h}\frac{1}{1+\theta}s_{t-j}|\{\omega_{t-j}\}_{j=0}^{\infty}\right] = \sum_{j=0}^{\infty} X_{j+h}\frac{1}{1+\theta}\omega_{t-j} = \frac{1}{1+\theta}\mathbb{E}_t[X_{t+h}]$$

which in operator form is

$$\mathcal{E}_{NS} = \frac{1}{1+\theta}\mathcal{E}$$

with spectral radius

$$r(\mathcal{E}_{NS}) = r(\frac{1}{1+\theta}\mathcal{E}) = \frac{1}{1+\theta}$$

An eigenvalue λ of $\mathcal{E}_{NS,\theta}$ satisfies

$$\lambda y_t = \mathbb{E}_t^{NS,\theta}[y_{t+1}]$$

$$= \frac{1}{1+\theta}\mathbb{E}_t[y_{t+1}]$$

which is solved by eigenseries

$$y_t = \lambda(1+\theta)y_{t-1} + \omega_t$$

Such a series may not satisfy Sunspot Admissibility in general, but will for simply stable models where $|\lambda(1+\theta)| < 1$ so that y_t is stationary. $\theta > 0$ so the condition can be rewritten as $|\lambda| < \frac{1}{1+\theta}$.

D.4.2 Beauty Contests

This section considers a "beauty contest," a game where forecasters face both incomplete information and strategic complementarity that warps their forecasts. Dynamic beauty contests further warp the forecasts made in models with noisy signals. Woodford (2003) is the classic example of this type of model.

When forecasting, agents would like to use the true values of ω_t, but can only use noisy signals thereof, as in Section D.4.1. Agent i receives the noisy public signal given by equation (38) as well as a noisy private signal p_{it} given by

$$p_{it} = \omega_t + \xi_{it}$$

where the noise term $\xi_{it} \sim N(0, \sigma_\xi^2)$ is i.i.d. across agents.

Agents literally have rational expectations, but do not make rational forecasts. Rather, they care about how their own forecasts compare to the population average, and in equilibrium appear to feature a behavioral bias. They forecast by

$$\mathbb{E}_{it}^{BC}[X_{t+h}] = \sum_{j=0}^{\infty} X_{j+h} \mathbb{E}_{it}^{BC}[\omega_{t-j}]$$

Where the backcast $\mathbb{E}_{it}^{BC}[\omega_{t-j}]$ is given by

$$\mathbb{E}_{it}^{BC}[\omega_{t-j}] = (1 - \theta^{BC})\mathbb{E}_{it}[\omega_{t-j}] + \theta^{BC}\mathbb{E}_{it}[\bar{\mathbb{E}}_t^{BC}[\omega_{t-j}]]$$

The rational expectation conditional on the two signals is

$$\mathbb{E}[\omega_t | p_{it}, s_t] = \frac{\theta}{\theta + \sigma_\xi^2 + \theta\sigma_\xi^2} p_{it} + \frac{\sigma_\xi^2}{\theta + \sigma_\xi^2 + \theta\sigma_\xi^2} s_t$$

or

$$\mathbb{E}[\omega_t | p_{it}, s_t] = b_p p_{it} + b_s s_t$$

Solving the beauty contest requires finding the unknown coefficients a_p and a_s in $\mathbb{E}_{i,t+j}^{BC}[\omega_t] = a_p p_{it} + a_s s_t$. Plug the rational expectation into the backcast:

$$\mathbb{E}_{i,t+j}^{BC}[\omega_t] = (1 - \alpha)(b_p p_{it} + b_s s_t) + \alpha \mathbb{E}_{i,t+j}[a_p \omega_t + a_s s_t]$$

$$a_p p_{it} + a_s s_t = (1 - \alpha + \alpha a_p)(b_p p_{it} + b_s s_t) + \alpha a_s s_t$$

collect coefficients:

$$a_p = (1 - \alpha + \alpha a_p) b_p$$

$$\implies a_p = \frac{(1 - \alpha) b_p}{1 - \alpha b_p}$$

$$a_s = (1 - \alpha + \alpha a_p) b_s + \alpha a_s$$

$$\implies a_s = b_s + \frac{\alpha}{1 - \alpha} a_p b_s$$

$$= \frac{b_s}{1 - \alpha b_p}$$

The average expectation is

$$\bar{\mathbb{E}}^{BC}_{t+j}[\omega_t] = a_p w_t + a_s s_t$$

so the projection onto w_t is

$$a_p w_t + a_s w_t = \left(\frac{(1-\alpha)b_p}{1-\alpha b_p} + \frac{b_s}{1-\alpha b_p} \right) w_t$$

$$= \frac{(1-\alpha)b_p + b_s}{1-\alpha b_p} w_t$$

$$= \frac{(1-\alpha)\theta + \sigma_\xi^2}{(1-\alpha)\theta + \sigma_\xi^2 + \theta\sigma_\xi^2} w_t$$

Or, to match Huo and Pedroni (2020) intuitive finding that this is just a rescaling of the variance of the private signal noise:

$$= \frac{\theta + \sigma_\xi^2/(1-\alpha)}{\theta + \sigma_\xi^2/(1-\alpha) + \theta\sigma_\xi^2/(1-\alpha)} w_t$$

$$= \frac{\theta + \phi}{\theta + \phi + \theta\phi} w_t$$

where $\phi \equiv \sigma_\xi^2/(1-\alpha)$, which may be positive or negative. Thus the behavioral expectation is written as

$$\mathbb{E}_t^{BC,\theta,\phi}[y_{t+1}] = \frac{\theta + \phi}{\theta + \phi + \theta\phi} \mathbb{E}_t[y_{t+1}]$$

which is the *average* expectation in the motivating beauty contest.

The operator representation is

$$\mathcal{E}_{BC} = \frac{\theta + \phi}{\theta + \phi + \theta\phi} \mathcal{E}$$

with spectral radius

$$r(\mathcal{E}_{BC}) = r(\frac{\theta + \phi}{\theta + \phi + \theta\phi} \mathcal{E}) = \frac{\theta + \phi}{\theta + \phi + \theta\phi}$$

An eigenvalue λ of $\mathcal{E}_{NS,\theta}$ satisfies

$$\lambda y_t = \mathbb{E}_t^{BC,\theta,\phi}[y_{t+1}]$$

$$= \frac{\theta + \phi}{\theta + \phi + \theta\phi}\mathbb{E}_t[y_{t+1}]$$

which is solved by eigenseries

$$y_t = \lambda \frac{\theta + \phi + \theta\phi}{\theta + \phi} y_{t-1} + \omega_t$$

Such a series may not satisfy Sunspot Admissibility in general, but will for simply stable models where $|\lambda \frac{\theta+\phi+\theta\phi}{\theta+\phi}| < 1$ so that y_t is stationary. ϕ may be negative, so the condition can be rewritten as $|\lambda| < |\frac{\theta+\phi}{\theta+\phi+\theta\phi}|$.

D.4.3 Signal Precision Overconfidence

This form of expectations marries incomplete information with a behavioral bias. Agents receive the noisy signal (38), but misperceive how noisy it is. The noise term is distributed $\nu_t \sim N(0, \theta)$, but agents mistakenly believe it is distributed $\nu_t \sim N(0, \theta/\phi)$. Typically, agents are considered to be overconfident about the signal's precision so that $\phi > 1$, but any positive value is allowable.[20]

Similar to Section D.4.1, agents backcast the shocks ω_t by

$$\mathbb{E}^{SPO}[\omega_t | s_t] = \frac{1}{1 + \theta/\phi} s_t$$

so the coefficient on ω_t alone is also $\frac{1}{1+\theta/\phi}$. Therefore the expectation operator is

$$\mathcal{E}_{SPO} = \frac{1}{1 + \theta/\phi}\mathcal{E}$$

with spectral radius

$$r(\mathcal{E}_{SPO}) = r(\frac{1}{1 + \theta/\phi}\mathcal{E}) = \frac{\phi}{\phi + \theta}$$

[20] A large literature following Odean (1998), Daniel, Hirshleifer, and Subrahmanyam (1998), Daniel, Hirshleifer, and Subrahmanyam (2001) and others study agents who are overconfident about the precision of their private signals, but also observe public signals. Huo and Pedroni (2020) show that beauty contest models with both types of signals can be re-parameterized as a model where agents only observe private signals, as in this section.

Noisy signals lower the spectral radius relative to full information, but overconfidence raises the spectral radius relative to the rational expectations case of Section D.4.1.

An eigenvalue λ of $\mathcal{E}_{SPO,\theta,\phi}$ satisfies

$$\lambda y_t = \mathbb{E}_t^{SPO,\theta,\phi}[y_{t+1}]$$

$$= \frac{\phi}{\phi+\theta}\mathbb{E}_t[y_{t+1}]$$

which is solved by eigenseries

$$y_t = \lambda\frac{\theta+\phi}{\phi}y_{t-1} + \omega_t$$

Such a series may not satisfy Sunspot Admissibility in general, but will for simply stable models where $|\lambda\frac{\theta+\phi}{\phi}| < 1$ so that y_t is stationary. $\theta > 0$ and $\phi > 0$, so the condition can be rewritten as $|\lambda| < \frac{\phi}{\theta+\phi}$.