Student Solutions Manual for

Real Analysis and Foundations

Fourth Edition

by Steven G. Krantz

Preface

This Manual contains the solutions to selected exercises in the book *Real Analysis and Foundations* by Steven G. Krantz, hereinafter referred to as "the text."

The problems solved here have been chosen with the intent of covering the most significant ones, the ones that might require techniques not explicitly presented in the text, or the ones that are not easily found elsewhere.

The solutions are usually presented in detail, following the pattern in the text. Where appropriate, only a sketch of a solution may be presented. Our goal is to illustrate the underlying ideas in order to help the student to develop his or her own mathematical intuition.

Notation and references as well as the results used to solve the problems are taken directly from the text.

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Chapter 1

Number Systems

1.1 The Real Numbers

1. The set $(0, 1]$ contains its least upper bound 1 but not its greatest lower bound 0. The set $[0, 1)$ contains its greatest lower bound 0 but not its least upper bound 1.

3. We know that $\alpha \geq a$ for every element $a \in A$. Thus $-\alpha \leq -a$ for every element $a \in A$ hence $-\alpha \leq b$ for every $b \in B$. If $b' > -\alpha$ is a lower bound for B then $-b' < \alpha$ is an upper bound for A, and that is impossible. Hence $-\alpha$ is the greatest lower bound for B.

Likewise, suppose that β is a greatest lower bound for A. Define $B = \{-a : a \in A\}$. We know that $\beta \leq a$ for every element $a \in A$. Thus $-\beta \geq -a$ for every element $a \in A$ hence $-\beta \geq b$ for every $b \in B$. If $b' < -\beta$ is an upper bound for B then $-b' > \beta$ is a lower bound for A, and that is impossible. Hence $-\beta$ is the least upper bound for B.

5. We shall treat the least upper bound. Let α be the least upper bound for the set S. Suppose that α' is another least upper bound. It $\alpha' > \alpha$ then α' cannot be the least upper bound. If $\alpha' < \alpha$ then α cannot be the least upper bound. So α' must equal α.

7. Let x and y be real numbers. We know that

$$(x + y)^2 = x^2 + 2xy + y^2 \leq |x|^2 + 2|x||y| + |y|^2 .$$

Taking square roots of both sides yields

$$|x + y| \leq |x| + |y| \, .$$

9. We treat commutativity. According to the definition in the text, we add two cuts $\mathcal{C}$ and $\mathcal{D}$ by

$$\mathcal{C} + \mathcal{D} = \{c + d : c \in \mathcal{C}, d \in \mathcal{D}\} \, .$$

But this equals

$$\{d + c : c \in \mathcal{C}, d \in \mathcal{D}\}$$

and that equals $\mathcal{D} + \mathcal{C}$.

11. Consider the set of all numbers of the form

$$\frac{j}{k\sqrt{2}}$$

for j, k relatively prime natural numbers and $j < k$. Then certainly each of these numbers lies between 0 and 1 and each is irrational. Furthermore, there are countably many of them.

* **13.** Notice that if $n - k\lambda = m - \ell\lambda$ then $(n - m) = (k - \ell)\lambda$. It would follow that λ is rational unless $n = m$ and $k = \ell$. So the numbers $n - k\lambda$ are all distinct.

Now let $\epsilon > 0$ and choose an positive integer N so large that $\lambda/N < \epsilon$. Consider $\varphi(1)$, $\varphi(2)$, $\ldots$, $\varphi(N)$. These numbers are all distinct, and lie in the interval $[0, \lambda]$. So two of them are distance not more than $\lambda/N < \epsilon$ apart. Thus $|(n_1 - k_1\lambda) - (n_2 - k_2\lambda)| < \epsilon$ or $|(n_1 - n_2) - (k_1 - k_2)\lambda| < \epsilon$. Let us abbreviate this as $|m - p\lambda| < \epsilon$.

It follows then that the numbers

$$(m - p\lambda), (2m - 2p\lambda), (3m - 3p\lambda), \ldots$$

are less than ϵ apart and fill up the interval $[0, \lambda]$. That is the definition of density.

1.2 The Complex Numbers

1. We calculate that

$$z \cdot \frac{\overline{z}}{|z|^2} = \frac{z \cdot \overline{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1\,.$$

So $\overline{z}/|z|^2$ is the multiplicative inverse of z.

3. Write

$$1 + i = \sqrt{2} e^{i\pi/4}\,.$$

We seek a complex number $z = re^{i\theta}$ such that

$$z^3 = r^3 e^{3i\theta} = (re^{i\theta})^3 = \sqrt{2} e^{i\pi/4}\,.$$

It follows that $r = 2^{1/6}$ and $\theta = \pi/12$. So we have found the cube root

$$c_1 = 2^{1/6} e^{i\pi/12}\,.$$

Now we may repeat this process with $\sqrt{2}e^{i\pi/4}$ replaced by $\sqrt{2}e^{i9\pi/4}$. We find the second cube root

$$c_2 = 2^{1/6} e^{i9\pi/12}\,.$$

Repeating the process a third time with $\sqrt{2}e^{i\pi/4}$ replaced by $\sqrt{2}e^{i17\pi/4}$, we find the third cube root

$$c_3 = 2^{1/6} e^{i17\pi/12}\,.$$

5. We see that

$$\phi(x + x') = (x + x') + i0 = (x + i0) + (x' + i0) = \phi(x) + \phi(x')\,.$$

Also

$$\phi(x \cdot x') = (x \cdot x') + i0 = (x + i0) \cdot (x' + i0) = \phi(x) \cdot \phi(x')\,.$$

7. Let
$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k$$
be a polynomial with real coefficients a_j. If α is a root of this polynomial then
$$p(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_k\alpha^k = 0\,.$$
Conjugating this equation gives
$$p(\alpha) = a_0 + a_1\overline{\alpha} + a_2\overline{\alpha}^2 + \cdots + a_k\overline{\alpha}^k = 0\,.$$
Hence $\overline{\alpha}$ is a root of the polynomial p. We see then that roots of p occur in conjugate pairs.

9. The function $\varphi(x) = x + i0$ from $\mathbb{R}$ to $\mathbb{C}$ is one-to-one. Therefore
$$\text{card}(\mathbb{R}) \leq \text{card}(\mathbb{C})\,.$$
Since the reals are uncountable, we may conclude that the complex numbers are uncountable.

11. The defining condition measures the sum of the distance of z to $1 + i0$ plus the distance of z to $-1 + i0$. If z is not on the x-axis then $|z - 1| + |z + 1| > 2$ (by the triangle inequality). If z is on the x axis but less than -1 or greater than 1 then $|z - 1| + |z + 1| > 2$. So the only z that satisfy $|z - 1| + |z + 1| > 2$ are those elements of the x-axis that are between -1 and 1 inclusive.

15. The set of all complex numbers with rational real part contains the set of all complex numbers of the form $0 + yi$, where y is any real number. This latter set is plainly uncountable, so the set of complex number with rational real part is also uncountable.

17. The set $S = \{z \in \mathbb{C} : |z| = 1\}$ can be identified with $T = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$. The set T can be identified with the interval $[0, 2\pi)$, and that interval is certainly an uncountable set. Hence S is uncountable.

19. Let p be a polynomial of degree $k \geq 1$ and let α_1 be a root of p. So $p(\alpha) = 0$. Now let us think about dividing $p(z)$ by $(z - \alpha_1)$. By the Euclidean algorithm,
$$p(z) = (z - \alpha_1) \cdot q_1(z) + r_1(z)\,. \qquad (*)$$

Here q_1 is the "quotient" and r_1 is the "remainder." The quotient will have degree $k-1$ and the remainder will have degree *less* than the degree of $z-\alpha_1$. In other words, the remainder will have degree 0—which means that it is constant. Plug the value $z=\alpha_1$ into the equation $(*)$. We obtain

$$0 = 0 + r_1 \, .$$

Hence the remainder, the constant r_1, is 0.

If $k=1$ then the process stops here. If $k>1$ then q_1 has degree $k-1 \geq 1$ and we may apply the Fundamental Theorem of Algebra to q_1 to find a root α_2. Repeating the argument above, we divide $(z-\alpha_2)$ into q_1 using the Euclidean algorithm. We find that it divides in evenly, producing a new quotient q_2.

This process can be repeated $k-2$ more times to produce a total of k roots of the polynomial p.

Chapter 2

Sequences

2.1 Convergence of Sequences

1. The answer is no. We can even construct a sequence with arbitrarily long repetitive strings that has subsequences converging to *any* real number α. Indeed, order $\mathbb{Q}$ into a sequence $\{q_n\}$. Consider the following sequence

$$\{q_1, q_2, q_2, q_1, q_1, q_1, q_2, q_2, q_2, q_2, q_3, q_3, q_3, q_3, q_3, q_1, q_1, q_1, q_1, q_1, q_1, \cdots\}.$$

In this way we have repeated each rational number infinitely many times, and with arbitrarily long strings. From the above sequence we can find subsequences that converge to any real number.

5. We know that

$$\int_0^1 \frac{dt}{1+t^2} = \mathrm{Tan}^{-1}(t)\Big|_0^1 = \frac{\pi}{4}.$$

As we know from calculus (and shall learn in greater detail in Chapter 7 of the present text), the integral on the left can be approximated by its Riemann sums. So we obtain

$$\sum_{j=0}^{k} f(s_j)\Delta x_j \approx \frac{\pi}{4}.$$

Here $f(t) = 1/(1+t^2)$. Since the sum on the left can be written out explicitly, this gives a means of calculating π to any desired degree of accuracy.

7. Let $\epsilon > 0$. Choose an integer J so large that $j > J$ implies that $|a_j - \alpha| < \epsilon$. Also choose an integer K so large that $j > K$ implies that $|c_j - \alpha| < \epsilon$. Let $M = \max\{J, K\}$. Then, for $j > M$, we see that

$$\alpha - \epsilon < a_j \le b_j \le c_j < \alpha + \epsilon \, .$$

In other words,

$$|b_j - \alpha| < \epsilon \, .$$

But this says that $\lim_{j\to\infty} b_j = \alpha$.

9. The sequence

$$a_j = \pi + \frac{1}{j} \, , \quad j = 1, 2, \ldots$$

is decreasing and certainly converges to π.

11. If the assertion were not true then the sequence $\{a_j\}$ does not converge. So, for any $\epsilon > 0$ there exist arbitarily large j so that $|a_j - \alpha| > \epsilon$. Thus we may choose $j_1 < j_2 < \cdots$ so that $|a_{j_k} - \alpha| > \epsilon$. This says that the subsequence $\{a_{j_k}\}$ does not converge to α. Nor does it have a subsequence that converges to α. That is a contradiction.

2.2 Subsequences

1. Let $a_1 \ge a_2 \ge \cdots$ be a decreasing sequence that is bounded below by some number M. Of course the sequence is bounded above by a_1. So the sequence is bounded. By the Bolzano-Weierstrass theorem, there is a subsequence $\{a_{j_k}\}$ that converges to a limit α.

Let $\epsilon > 0$. Choose $K > 0$ so that, when $k \ge K$, $|a_{j_k} - \alpha| < \epsilon$. Then, when $j > j_K$,

$$\alpha - \epsilon < a_j \le a_{j_K} < \alpha + \epsilon \, .$$

Thus

$$|a_j - \alpha| < \alpha \, .$$

So the sequence converges to α.

3. Suppose that $\{a_\}$ has a subsequence diverging to $+\infty$. If in fact $\{a_j\}$ converges to some finite real number α, then every subsequence converges to α. But that is a contradiction.

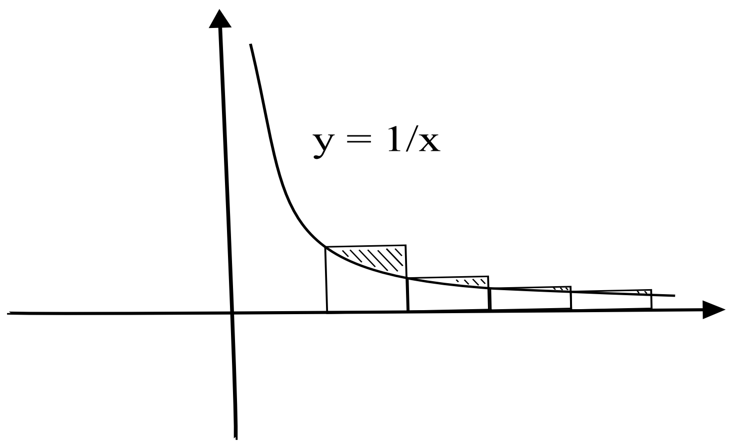

Figure 2.1: Sum of shaded regions is $1 + 1/2 + \cdots 1/j - \log j$.

5. Consider Figure 2.1.

The sum of the areas of the four shaded regions is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \log j\,,$$

where of course we use the natural logarithm. All four of these shaded regions may be slid to the left so that they lie in the first, large box. And they do not overlap. This assertion is true not just for the first four summands but for any number of summands. So we see that the value of

$$\lim_{j\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}\right) - \log j$$

is not greater that $1 \times 1 = 1$. In particular, the sequence is increasing and bounded above. So it converges.

7. Similar to the solution of Exercise 13 in Section 1.1 above.

9. Define the sequence a_j by

$$0, 0, 1, 0, 1, 1/2, 0, 1, 1/2, 1/3, 0, 1, 1/2, 1/3, 1/4, \ldots\,.$$

Then, given an element $1/j$ in S, we may simply choose the subsequence

$$1/j, 1/j, 1/j, \ldots$$

from the sequence a_j to converge to $1/j$. And it is clear that the subsequences of a_j have no other limits.

2.3 Lim sup and Lim inf

1. Consider the sequence

$$1, -1, 1, -1, 5, -5, 1, -1, 1, -1, \ldots .$$

Then, considered as a sequence, the limsup is plainly 1. But the supremum of the set of numbers listed above is 5. Also the liminf is -1. But the infimum of the set of numbers listed above is -5.

What is true is that

$$\limsup a_j \le \sup\{a_j\}$$

and

$$\liminf a_j \ge \inf\{a_j\} .$$

We shall not prove these two inequalities here.

3. Let $\alpha = \limsup a_j$. Then there is a subsequence $\{a_{j_k}\}$ that converges to α. But then $\{-a_{j_k}\}$ converges to $-\alpha$. If there is some other subsequence $\{-a_{j_\ell}\}$ that converges to some number $\beta < -\alpha$ then $\{a_{j_\ell}\}$ would converge to $-\beta > \alpha$. And that is impossible. Hence $-\alpha$ is the liminf of $\{-a_j\}$.

A similar argument applies to $\gamma = \liminf a_j$ and the consideration of $\{-a_j\}$.

5. Consider the sequence

$$a, b, a, b, a, b, a, b, \ldots .$$

Then clearly the limsup of this sequence is equal to b and the liminf of this sequence is equal to a.

9. The limsup is defined to be the limit of the sequence $b_j = \sup\{a_j, a_{j+1}, a_{j+2}, \ldots\}$. Clearly $b_j \ge a_j$. Therefore

$$\lim_{j\to\infty} b_j = \lim_{k\to\infty} b_{j_k} \ge \lim_{k\to\infty} a_{j_k} .$$

So

$$\lim_{k\to\infty} a_{j_k} \le \limsup a_j .$$

A similar argument shows that

$$\lim_{k\to\infty} a_{j_k} \ge \liminf a_j .$$

11. Let $\{a_{j_\ell}\}$ be any subsequence of the given sequence. Define $b_{j_\ell} = \sup\{a_{j_\ell}, a_{j_{\ell+1}}, \ldots\}$. Then

$$b_{j_\ell} \geq a_{j_\ell}$$

so

$$\limsup_{\ell \to \infty} b_{j_\ell} \geq \limsup \ell \to \infty a_{j_\ell}$$

so that

$$\limsup a_{j_\ell} \geq \limsup a_{j_\ell} .$$

A similar argument applies to the liminf.

* **13.** The numbers $\{\sin j\}$ are dense in the interval $[-1, 1]$ (see Exercise 7 of Section 2.2). Thus, given $\epsilon > 0$, there is an integer j so that $|\sin j - 1| < \epsilon$. But then

$$|\sin j|^{\sin j} > (1 - \epsilon)^{1-\epsilon} .$$

It follows that

$$\limsup |\sin j|^{\sin j} = 1 .$$

A similar argument shows that

$$\liminf |\sin j|^{\sin j} = (1/e)^{1/e} .$$

2.4 Some Special Sequences

1. Let $r = p/q = m/n$ be two representations of the rational number r. Recall that for any real α, the number α^r is defined as the real number β for which

$$\alpha^m = \beta^n.$$

Let β' satisfy

$$\alpha^p = \beta'^q.$$

We want to show that $\beta = \beta'$. we have

$$\begin{aligned} \beta^{n \cdot q} &= \alpha^{m \cdot q} \\ &= \alpha^{p \cdot n} \\ &= \beta'^{q \cdot n}. \end{aligned}$$

By the uniqueness of the $(n \cdot q)^{th}$ root of a real number it follows that

$$\beta = \beta',$$

proving the desired equality. The second equality follows in the same way. Let

$$\alpha = \gamma^n.$$

Then

$$\alpha^m = \gamma^{n \cdot m}.$$

Therefore, if we take the n^{th} root on both sides of the above inequality, we obtain

$$\gamma^m = (\alpha^m)^{1/n}.$$

Recall that γ is the n^{th} root of α. Then we find that

$$(\alpha^{1/n})^m = (\alpha^m)^{1/n}.$$

Using similar arguments, one can show that for all real numbers α and β and $q \in \mathbb{Q}$

$$(\alpha \cdot \beta)^q = \alpha^q \cdot \beta^q.$$

Finally, let α, β, and γ be positive real numbers. Then

$$\begin{aligned}
(\alpha \cdot \beta)^\gamma &= \sup\{(\alpha \cdot \beta)^q : q \in \mathbb{Q}, q \leq \gamma\} \\
&= \sup\{\alpha^q \beta^q : q \in \mathbb{Q}, q \leq \gamma\} \\
&= \sup\{\alpha^q : q \in \mathbb{Q}, q \leq \gamma\} \cdot \sup\{\beta^q : q \in \mathbb{Q},\ q \leq \gamma\} \\
&= \alpha^\gamma \cdot \beta^\gamma.
\end{aligned}$$

3. We write

$$\frac{j^j}{(2j)!} = \frac{1}{1 \cdot 2 \cdots (j-1) \cdot (j)} \cdot \frac{j \cdot j \cdot \cdots \cdot j}{(j+1) \cdot (j+2) \cdot \cdots \cdot 2j}\,.$$

Now the second fraction is clearly bounded by 1, while the first fraction is bounded by $1/((j-1)j)$. Altogether then,

$$0 \leq \frac{j^j}{(2j)!} \leq \frac{1}{j^2 - j}\,.$$

The righthand side clearly tends to 0. So

$$\lim_{j \to \infty} \frac{j^j}{(2j)!} = 0\,.$$

7. Use a generating function as in the solution of Exercise 6 above.

* **9.** Notice that

$$\left(1+\frac{1}{j^2}\right) \leq \exp(1/j^2)$$

(just examine the power series expansion for the exponential function). Thus

$$\begin{aligned} a_j &= \left(1+\frac{1}{1^2}\right)\cdot\left(1+\frac{1}{2^2}\right)\cdot\left(1+\frac{1}{1^3}\right)\cdot\cdots\cdot\left(1+\frac{1}{j^2}\right) \\ &\leq \exp(1/1^2)\cdot\exp(1/2^2)\cdot\exp(1/3^2)\cdot\cdots\cdot\exp(1/j^2) \\ &= \exp(1/1^2+1/2^2+1/3^2+\cdots+1/j^2)\,. \end{aligned}$$

Of course the series in the exponent on the right converges. So we may conclude that the infinite product converges.

Chapter 3

Series of Numbers

3.1 Convergence of Series

1. **(a)** Converges by the Ratio Test.

(b) Diverges by comparison with $\sum_j 1/j$.

(c) Converges by the Ratio Test.

(d) Converges by the Alternating Series Test.

(e) Diverges by comparison with $\sum_j 1/j$.

(f) Converges by comparison with $\sum_j 1/j^2$.

(g) Converges by the Root Test.

(h) Converges by the Cauchy Condensation Test.

(i) Diverges by comparison with $\sum_j 1/j$.

(j) Converges by the Cauchy Condensation Test.

3. Since $\sum_j b_j$ converges, then $b_j \to 0$. Thus, for j sufficiently large, $0 < b_j < 1/2$. But then

$$\frac{1}{1+b_j} \geq \frac{2}{3}.$$

So the series diverges by the Comparison Test.

5. FALSE. Let $a_j = (j+1)^2$. Then $a_j > 1$ for all $j = 1, 2, \ldots$. And $\sum_j 1/a_j$ converges. But $\sum_j a_j$ diverges.

7. We will prove the more general fact that if $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ are convergent series of positive numbers, then

$$\left(\sum_{j=0}^{\infty} a_j b_j\right)^2 \leq \left(\sum_{j=0}^{\infty} a_j^2\right)\left(\sum_{j=0}^{\infty} b_j^2\right).$$

First, recall the Cauchy product of series:

$$\begin{aligned}\left(\sum_{j=0}^{\infty} a_j b_j\right)\left(\sum_{j=0}^{\infty} a_j b_j\right) &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_{n-j} b_{n-j} a_j b_j\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left[a_{n-j}^2 b_j^2 + a_j^2 b_{n-j}^2\right].\end{aligned}$$

To prove the inequality observe that, for each j,

$$2a_{n-j}a_j b_{n-j} b_j \leq a_{n-j}^2 b_j^2 + a_j^2 b_{n-j}^2$$

because of the inequality

$$2cd \leq c^2 + d^2.$$

Finally notice that

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n} a_{n-j}^2 b_j^2 = \left(\sum_{j=0}^{\infty} a_j^2\right)\left(\sum_{j=0}^{\infty} b_j^2\right)$$

and

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n} a_j^2 b_{n-j}^2 = \left(\sum_{j=0}^{\infty} a_j^2\right)\left(\sum_{j=0}^{\infty} b_j^2\right)$$

by the Cauchy product formula again. The proof of the inequality is complete.

In order to finish the exercise notice that for

$$\alpha > 1/2$$

the series

$$\sum_{j=1}^{\infty} \frac{1}{(j^\alpha)^2}$$

is convergent and so is

$$\sum_{j=1}^{\infty} b_j .$$

9. Of course $b_j/j^2 \leq b_j$, so that $\sum_j b_j/j^2$ converges by the Comparison Test.

11. If $\alpha > 1$, then we have convergence by the Comparison Test. If $\alpha = 1$, then we have convergence provided that $\beta > 1$ (by the Cauchy Condensation Test). Otherwise we have divergence.

3.2 Elementary Convergence Tests

1. If $\sum_{j=1}^{\infty} b_j$ converges than $b_j \longrightarrow 0$. Then there exists $N > 0$ such that for $j \geq N, b_j \leq 1$. If p is a polynomial with no constant term, then ther exists x_0 and a constant $C > 0$ such that for $0 \leq x \leq x_0$,

$$p(x) \leq C \cdot x .$$

Indeed, if $p(x) = a_1 x + a_2 x^2 + \cdots + a_n x^n$, then $p(x) \leq 2a_1 x$ for x close enough to 0. Since $\sum_{j=1}^{\infty} b_j$ converges, $b_j \to 0$. Then there exists N such that, for $j > N$, $b_j < x_0$. Thus,

$$p(b_j) < C b_j .$$

By the Comparison Test we are done.

3. For the first series, the Root Test gives

$$|1/j|^{1/j} \to 1 ,$$

which is inconclusive. The Ratio Test gives

$$\frac{1/(j+1)}{1/j} = \frac{j}{j+1} \to 1 ,$$

which is inconclusive.

For the second series, the Root Test gives

$$|1/j^2|^{1/j} \to 1 ,$$

which is inconclusive. The Ratio Test gives

$$\frac{1/(j+1)^2}{1/j^2} = \frac{j^2}{(j+1)^2} \to 1\,,$$

which is inconclusive.

However we know that $\sum_j 1/j$ diverges and $\sum_j 1/j^2$ converges.

5. By our hypothesis, there is a number $0 < \beta < 1$ and an integer $N > 1$ such that, for all $j > N$, it holds that

$$\left|\frac{a_{j+1}}{a_j}\right| < \beta$$

or

$$|a_{j+1}| < \beta |a_j|\,.$$

We may apply the last line with j replaced by $j+1$ to obtain

$$|a_{j+2}| < \beta |a_{j+1}|$$

hence

$$|a_{j+2}| < \beta^2 |a_j|\,.$$

Continuing, we finally obtain

$$|a_{j+k}| < \beta^k |a_j|\,.$$

Thus we see that the series $\sum_{j=N+1}^{\infty} |a_j|$ is majorized by the convergent geometric series $|a_N| \cdot \sum_{j=N+1}^{\infty} \beta^{j-N}$. Hence the series converges.

7. Assume that all the a_j are positive. Let $\liminf a_{j+1}/a_j = a$. Plainly $a \ge 0$. If $a = 0$ then there is nothing to prove, so suppose that $a > 0$. Let $0 < \alpha < a$. Certainly there is a positive integer N so that

$$\frac{a_{j+1}}{a_j} > \alpha \qquad (*)$$

when $j \ge N$. For $M > N$, multiply together the inequalities $(*)$ for $j = N, N+1, \ldots, M-2, M-1$ to obtain

$$\frac{a_M}{a_N} > \alpha^{M-N}$$

hence

$$\sqrt[M]{a_M} > \alpha \cdot \sqrt[M]{a_N \alpha^{-N}} \,. \qquad (**)$$

Taking the liminf as $M \to \infty$, we find that

$$\liminf_{M\to\infty} \sqrt[M]{a_M} \geq \liminf_{M\to\infty} \alpha \cdot \sqrt[M]{a_N \alpha^{-N}} = \alpha \,.$$

Since α was an arbitrary positive number less than a, we conclude that

$$\liminf_{M\to\infty} \sqrt[M]{a_M} \geq \liminf \frac{a_{M+1}}{a_M} \,.$$

A similar argument shows that

$$\limsup_{M\to\infty} \sqrt[M]{a_M} \leq \limsup \frac{a_{M+1}}{a_M} \,.$$

This shows that if a series passes the Ratio Test then it also passes the Root Test.

9. TRUE. Since $\sum_j b_j$ converges, the terms b_j tend to zero. Hence, for j sufficiently large, $0 < b_j < 1$. Therefore

$$a_j b_j < a_j \,.$$

So the convergence of $\sum_j a_j b_j$ follows from the Comparison Test.

3.3 Advanced Convergence Tests

1. We note that

$$\frac{b_j}{1 - b_j} \leq \frac{b_j}{1/2} = 2b_j \,.$$

So the series $\sum_j b_j/(1 - b_j)$ converges by the Comparison Test.

3. Let $f(x) = \sqrt{x}$. Applying the Mean Value Theorem (see Section 6.2, or consult your calculus book), we see that

$$f(2j + 3) - f(2j) = ((2j + 3) - 2j) \cdot f'(\xi)$$

for some $2j \leq \xi \leq 2j + 3$. Since $f'(\xi) = 1/(2\sqrt{\xi})$, we find that

$$|f(2j + 3) - f(2j)| \leq \frac{3}{2\sqrt{2}j^{1/2}} \,.$$

Thus the summands of the series are majorized by

$$\left(\frac{(2j+3)^{1/2} - (2j)^{1/2}}{j^{3/4}} \right) \le \frac{3}{2\sqrt{2} j^{5/4}} .$$

The sum of the terms on the right converges. So the original series is absolutely convergent.

5. Suppose that

$$p(j) = a_0 + a_1 j + a_2 j^2 + \cdots + a_k j^k$$

is a polynomial with integer coefficients.

(a) Suppose that p has the property that $p(j)$ is even for every integer j. If we add a term $a_{k+1} j^{k+1}$ to p in order to increase the degree of p by 1, then we are adding a term that alternates in parity (odd-even-odd-even etc. It follows that the new polynomial

$$\widetilde{p}(j) = a_0 + a_1 j + a_2 j^2 + \cdots + a_k j^k + a_{k+1} j^{k+1}$$

alternates parity.

(b) Suppose instead p has the property that $p(j)$ is odd for every integer j. If we add a term $a_{k+1} j^{k+1}$ to p in order to increase the degree of p by 1, then we are adding a term that alternates in parity (odd-even-odd-even etc. It follows that the new polynomial

$$\widetilde{p}(j) = a_0 + a_1 j + a_2 j^2 + \cdots + a_k j^k + a_{k+1} j^{k+1}$$

alternates parity.

(c) Suppose instead that p has the property that $p(j)$ alternates parity with increasing j. If we add a term $a_{k+1} j^{k+1}$ to p in order to increase the degree of p by 1, then we are adding a term that alternates in parity (odd-even-odd-even etc. It follows that the new polynomial

$$\widetilde{p}(j) = a_0 + a_1 j + a_2 j^2 + \cdots + a_k j^k + a_{k+1} j^{k+1}$$

is either always even or always odd.

This proves the required assertion about $p(j)$.

7. Refer to the solution of Exercise 7 in Section 3.1. Let $a_j = (\gamma_j)^{1/2}$ and $b_j = 1/j^\alpha$. Then we know that

$$\sum_j (\gamma_j)^{1/2} \cdot \frac{1}{j^\alpha} \le \left(\sum_j \gamma_j \right)^{1/2} \cdot \left(\sum_j \frac{1}{j^{2\alpha}} \right)^{1/2} .$$

As long as $\alpha > 1/2$, both series on the right converge. So the series on the left converges.

When $\alpha = 1/2$, consider the example of $\gamma_j = 1/(j \cdot (\log(j+1))^{1.1})$. Then $\sum_j \gamma_j$ converges but

$$\sum_j (\gamma_j)^{1/2} \cdot \frac{1}{j^{1/2}} = \sum_j \frac{1}{j \cdot (\log(j+1)^{0.55})} ,$$

and that series diverges.

* **9.** The series $\sum_j 1/j$ diverges. Moreover, it can be shown that the partial sum s_j has size $C \cdot \log(j+1)$. Thus $b_j/s_j \approx 1/(j \log j)$. And the series

$$\sum_j \frac{1}{j \log(j+1)}$$

diverges.

3.4 Some Special Series

1. We will do the sum of the first N perfect cubes. Let the inductive statement $P(N)$ be

The sum of the first N perfect cubes is

$$S_N = \frac{N^2(N+1)^2}{4} .$$

Clearly $P(1)$ is true since then $S_N = 1 \cdot 4/4 = 4$. Now assume that $P(j)$ has been proved. Thus

$$1^3 + 2^3 + \cdots + j^3 = \frac{j^2(j+1)^2}{4} .$$

We add $(j+1)^3$ to both sides. So

$$1^3 + 2^3 + \cdots + j^3 + (j+1)^3 = \frac{j^2(j+1)^2}{4} + (j+1)^3 .$$

Notice that the lefthand side is just the sum of the first $(j+1)$ perfect cubes. We use some algebra to simplify the righthand side:

$$\begin{aligned}
\frac{j^2(j+1)^2}{4} + (j+1)^3 &= \frac{j^4 + 2j^3 + j^2 + 4j^3 + 12j^2 + 12j + 4}{4} \\
&= \frac{j^4 + 6j^3 + 13j^2 + 12j + 4}{4} \\
&= \frac{(j^2 + 2j + 1) \cdot (j^2 + 4j + 4)}{4} \\
&= \frac{(j+1)^2 \cdot (j+2)^2}{4} .
\end{aligned}$$

Thus we have

$$1^3 + 2^3 + \cdots + j^3 + (j+1)^3 = \frac{(j+1)^2 \cdot (j+2)^2}{4} .$$

This is $P(j+1)$. So we have completed the induction and the formula is proved.

3. Notice that, no matter what the value of k,

$$\frac{1}{|\ln j|^k} \geq \frac{1}{j}$$

for j sufficiently large. By the Comparison Test, the series diverges.

5. Suppose that p is a polynomial of degree k. For j large, $p(j)$ is comparable to $C \cdot j^k$. So we should compare with the series $\sum_j e^{C \cdot j^k}$. If C is negative, then the series converges by the Root Test. If C is positive, then the terms of this series do not tend to 0. So the series diverges.

7. If both $\pi + e$ and $\pi - e$ are algebraic, then (according to Exercise 6) $(\pi + e) + (\pi - e) = 2\pi$ is algebraic. But that is false. So one of $\pi + e$ or $\pi - e$ must be transcendental.

3.5 Operations on Series

1. Let $a_j = 1/j^3$ and $b_j = 1/j^4$. We know that

$$\begin{aligned} c_m &= \sum_{j=0}^{m} a_j b_{m-j} \\ &= \sum_{j=0}^{m} \frac{1}{j^3} \cdot \frac{1}{(m-j)^4} \\ &\le \sum_{j=0}^{[m/2]} 1 \cdot \frac{1}{(m/2)^4} + \sum_{[m/2]}^{m} \frac{1}{(m/2)^3} \cdot 1 \\ &\approx \frac{m}{2} \cdot \frac{16}{m^4} + \frac{m}{2} \cdot \frac{8}{m^3} \\ &= \frac{8}{m^3} + \frac{4}{m^2} . \end{aligned}$$

Thus

$$\sum_m c_m \le \sum_m \frac{8}{m^3} + \sum_m \frac{4}{m^2} .$$

The two series on the right converge, hence $\sum_m c_m$ converges.

3. The safest way to consider the composition of two power series

$$A = \sum_{j=0}^{\infty} a_j x^j$$

and

$$B = \sum_j b_{j=0}^{\infty} x^j$$

is to consider their partials sums

$$S_N^A = \sum_{j=0}^{N} a_j x^j$$

and

$$S_N^B = \sum_{j=0}^{N} b_j x^j .$$

Then the composition $S_N^A \circ S_N^B$ makes perfect sense and we can consider the limit of the composition as $N \to \infty$.

5. The convergence follows from Exercise 7 in Section 3.1.

7. Let $A = \sum_j a_j$ be a convergent series and let c be a nonzero constant. Let $S_N = \sum_{j=1}^N a_j$ be the partial sums for the series A. Then cS_N are the partial sums for the series $\sum_j ca_j$. Let $\epsilon > 0$. Choose M so large that $|S_N - \alpha| < \epsilon/|c|$ when $N > M$.

For such N we have

$$|cS_N - c\alpha = |c||S_N - \alpha| < |c| \cdot \frac{|\epsilon|}{|c|} = \epsilon\,.$$

It follows that $cS_N \to c\alpha$, so the series $\sum_j ca_j$ converges to $c\alpha$.

* **9.** Let $A = \sum_{j=0}^\infty a_j x^j$ be a power series. Let $S_N = \sum_{j=0}^N a_j x^j$ be a partial sum. Then

$$e^{S_N} = e^{a_0} \cdot e^{a_1 x} \cdot e^{a_2 x^2} \cdot \cdots \cdot e^{a_N x^N}\,.$$

So we see that the exponential of a power series can be interpreted in the language of an infinite product.

Chapter 4

Basic Topology

4.1 Open and Closed Sets

1. Let $t \in T$. Then there is an $s \in S$ with $|t - s| < \epsilon$. Let $x \in (t - (\epsilon - |t - s|), t + (\epsilon - |t - s|))$. Then

$$|x - s| \leq |x - t| + |t - s| < [\epsilon - |t - s|] + |t - s| = \epsilon\,.$$

Hence $x \in T$. This shows that T is open.

3. The set $S = [0, 1)$ is not open and it is also not closed.

5. Let $X_j = [j, \infty)$. Then $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$ but $\cap_j X_j = \emptyset$.

7. Let $U_j = (-1/j, 1 + 1/j)$. Then $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ and $\cap_j U_j = [0, 1]$.

9. The set S is not open because if $q \in S$ and $\epsilon > 0$, then $(q - \epsilon, q + \epsilon)$ contains both rational and irrational points. The complement of S is the set of irrational numbers. It is not open because if $x \in {}^c S$ and $\epsilon > 0$ then $(x - \epsilon, x + \epsilon)$ contains both rationals and irrationals. So S is not closed.

* **11.** Let $x \notin \overline{S}$. If $s \in S$ then $|x - s| > 0$. If there exist $s_j \in S$ so that

$$|x - s_j| \to 0\,,$$

then let $\epsilon > 0$. If j is larger than some $N > 0$, then

$$|x - s_j| < \epsilon\,.$$

Thus if $j, k > N$, then

$$|s_j - s_k| \leq |s_j - x| + |x - s_k| < \epsilon + \epsilon\,.$$

We conclude that the sequence $\{s_j\}$ is Cauchy. Since $\overline{S}$ is closed, we may conclude that $\{s_j\}$ has a limit point $s^* \in \overline{S}$. And it follows that $|x - s^*| = 0$. But this means that $x \in \overline{S}$. And that is impossible. Thus x has a positive distance ϵ from S.

Now if $x, y \in \mathbb{R}$ and $s \in S$, then

$$|x - s| \leq |x - y| + |y - s|\,.$$

Taking the infimum on the left over all $s \in S$ yields

$$\text{dist}(x, S) \leq |x - y| + |y - s|\,.$$

Now choosing s to be very nearly the nearest point in S to y, we see that

$$\text{dist}(x, S) \leq |x - y| + \text{dist}(y, S)\,. \qquad (*)$$

Reversing the roles of x and y gives the similar inequality

$$\text{dist}(y, S) \leq |x - y| + \text{dist}(x, S)\,. \qquad (**)$$

Now combining $(*)$ and $(**)$ we find that

$$|\text{dist}(x, S) - \text{dist}(y, S)| \leq |x - y|\,.$$

4.2 Further Properties of Open and Closed Sets

1. The set $\overline{S}$ is the intersection of all closed sets that contain S. So certainly $\overline{S}$ contains S. If $x \notin \overline{S}$, then $x \notin E$ for some closed set E that contains S. Fix such an E. Then there is an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq {}^c E$. But $\overline{S} \subseteq E$ so $(x - \epsilon, x + \epsilon) \subseteq {}^c\overline{S}$. Thus $\overline{S}$ is closed.

If $x \in \overline{S} \setminus \overset{\circ}{S}$, then there is no $\epsilon > 0$ so that $(x - \epsilon, x + \epsilon) \subseteq S$. So $(x - \epsilon, x + \epsilon)$ intersects ${}^c S$. But $x \in \overline{S}$, so x intersects every closed set that contains S. If there were an $\epsilon > 0$ so that $(x - \epsilon, x + \epsilon) \cap S = \emptyset$, then $\mathbb{R} \setminus (x - \epsilon, x + \epsilon)$ would be a closed set containing S that does not have x as an element. That is a contradiction. Thus every open interval about x intersects both S and ${}^cS$. So $x \in \partial S$.

Conversely, if $x \in \partial S$, then every neighborhood of x intersects both S and ${}^cS$. So x is not in the interior of S. If E is a closed set containing S and $x \notin E$, then x could not be in the boundary of S. So $x \in \overline{S}$. Hence $\partial S \subseteq \overline{S} \setminus \overset{\circ}{S}$.

3. Let $E_j = [1/j, 1 - 1/j]$. Each E_j is closed, but $\cup_j E_j = (0, 1)$, which is open.

Let $E_j = [0, 1]$ for every j. Then each E_j is closed. Also $\cup_j E_j = [0, 1]$, which is closed.

Let $E_j = [0, 1 - 1/j]$. Then each E_j is closed, but $\cup_j E_j = [0, 1)$ is neither open nor closed.

5. Let $S \subseteq \mathbb{R}$ be any set. Let x lie in $\overset{\circ}{S}$. By definition, there is an $\epsilon > 0$ so that $(x - \epsilon, x + \epsilon) \subseteq S$. Now let $t \in (x - \epsilon, x + \epsilon)$. Let $\delta = \min\{((x + \epsilon) - t), t - (x - \epsilon)\}$. Let $s \in (t - \delta, t + \delta)$. Assume for simplicity that $t < x$. Then

$$|x - s| \leq |x - t| + |t - s| < (x - t) + (t - x + \epsilon) = \epsilon\,.$$

Therefore $s \in (x - \epsilon, x + \epsilon)$. We conclude that $(t - \delta, t + \delta) \subseteq (x - \epsilon, x + \epsilon) \subseteq S$. So $\overset{\circ}{S}$ is open.

If S is open then each $s \in S$ has an $\epsilon > 0$ so that $(s - \epsilon, s + \epsilon) \subseteq S$. Thus $S \subseteq \overset{\circ}{S}$. Also if $x \in \overset{\circ}{S}$ then there is a $\delta > 0$ so that $(x - \delta, x + \delta) \subseteq S$. So $x \in S$. Therefore, if S is open then $S = \overset{\circ}{S}$.

Conversely, if $S = \overset{\circ}{S}$, then each $x \in S$ has an $\epsilon > 0$ so that $(x - \epsilon, x + \epsilon) \subseteq S$. So S is open.

7. If $x \in \mathbb{R}$ and $\epsilon > 0$ then the interval $(x - \epsilon, x + \epsilon)$ contains both points of S and points of ${}^cS$. This shows that x is *not* an interior point of S, but that x *is* a boundary point of S. So every real is in the boundary of S and no real is in the interior of S. If E is a closed set that contains S then E must contain all the limit points of S so E must contain $\mathbb{R}$. Thus the closure of S is the entire real line.

* **9.** Let $S = \{(x, 1/x) : x \in \mathbb{R}, x > 0\}$. Certainly $S \subseteq \mathbb{R}^2$ is closed. But the projection to the x-axis is $\pi S = \{x \in \mathbb{R} : x > 0\}$ which is open and not closed.

4.3 Compact Sets

1. Let E be closed and K compact. Then K is closed and bounded. It follows that $E \cap K$ is closed. Also $E \cap K$ is bounded. So $E \cap K$ is compact.

3. If not then there exist points $k_j \in K$ and $\ell_j \in L$ so that $|k_j - \ell_j| \to 0$ as $j \to \infty$. By the compactness of K, we may choose a subsequence k_{j_m} that converges to a point $k_0 \in K$. We see that

$$|\ell_{j_m}| \leq |\ell_{j_m} - k_{j_m}| + |k_{j_m} - k_0| + |k_0| .$$

The first term on the right tends to 0, the second tends to 0, and the third is a fixed constant. So we see that $|\ell_{j_m}|$ is bounded. By Bolzano-Weierstrass, there is a convergent subsequence $\ell_{j_{m_n}}$ which converges to some point $\ell_0 \in L$.

But now it follows that

$$|k_0 - \ell_0| = \lim_{n\to\infty} |k_{j_{m_n}} - \ell_{j_{m_n}}| = 0 .$$

So $k_0 = \ell_0$. This contradicts the hypothesis that K and L are disjoint. It follows that there is a positive distance between K and L.

5. For each $k \in K$, choose a $\delta_k > 0$ so that the interval $(k - \delta_k, k + \delta_k)$ lies in some U_j. The intervals $(k - \delta_k/3, k + \delta_k/3)$ form an open cover of K. So there is a finite subcover

$$(k_1 - \delta_{k_1}/3, k_1 + \delta_{k_1}/3), (k_2 - \delta_{k_2}/3, k_2 + \delta_{k_2}/3), \ldots, (k_\ell - \delta_{k_\ell}/3, k_\ell + \delta_{k_\ell}/3) .$$

Now if p is any point of K, then p lies in some $(k_j - \delta_{k_j}/3, k_j + \delta_{k_j}/3)$ which in turn lies in some U_{k_j}. But then, by the triangle inequality, $(p - \delta_{k_j}/3, p + \delta_{k_j}/3)$ also lies in U_{k_j}.

7. The set $\widehat{K}$ will be bounded and closed, so it will certainly be compact.

9. The intersection of any number of closed sets is closed. And the intersection of any number of bounded sets is bounded. So the intersection of any number of compact sets is compact.

11. The open set $U = (0, \infty)$ is unbounded, so it plainly cannot be written as the decreasing intersection of compact sets.

4.4 The Cantor Set

1. There is nothing special about 5 in this problem. Let $0 < \lambda < 1$ be fixed (you can set $\lambda = 1/5$ if you like). Let us introduce a convention. If a set S is the union of finitely many disjoint intervals, we define the length of S to be the sum of the lengths of the intervals whose union is S. For the first step we remove the middle λ-part of the interval $[0, 1]$. This produces a set, call it S_1, that is the union of two intervals the sum of whose lengths is $1 - \lambda$, that is, the length of S_1 is $1 - \lambda$. From each of these intervals remove the middle λ -part. This produces a set, call it S_2, (a union of four intervals) whose length is

$$1 - \lambda - \lambda(1 - \lambda) = (1 - \lambda)^2.$$

Now continue this process. At the j^{th} step we have the set S_j of length $(1 - \lambda)^j$. The set S_{j+1} is obtained by removing the λ-part of it. Hence S_{j+1} has length

$$(1 - \lambda)^j - \lambda(1 - \lambda)^j = (1 - \lambda)^{j+1}.$$

Notice that we are always measuring the length of the set that we *save*. The Cantor-like set C is obtained as

$$C = \bigcap_{j=1}^{\infty} S_j.$$

Also notice that

$$S_j \supseteq S_{j+1} \qquad \text{for all} \quad j.$$

Then

$$\text{length}(S_j) \geq \text{length}(S_{j+1}).$$

But

$$\lim_{j\to\infty} \text{length}(S_j) = \lim_{j\to\infty} (1-\lambda)^j = 0.$$

We see that the length of the Cantor-like set is 0. The set C is compact (and non-empty) since it is the intersection of nested compact sets. Moreover, C is totally disconnected. Arguing as in the text, let $\delta = |x-y|$, and let j satisfy $\frac{(1-\lambda)^j}{2^j} < \lambda$. Notice that the set S_j is the union of 2^j intervals each of length $\frac{(1-\lambda)^j}{2^j}$. Since x, y both belong to S_j, they cannot lie in the same interval. Hence between x and y there exist numbers that are not in C. Thus, C is totally disconnected.

The proof that C is perfect is exactly as in the text for the case $\lambda = 3$.

3. Let $U = (-1, 0)$ and $V = (0, 1)$. Then U and V are disjoint open sets, but the distance between U and V is 0.

5. Each removed interval has two endpoints, and there are countably many removed intervals. So the total number of endpoints is countable. The Cantor set is uncountable. So the number of non-endpoints is uncountable.

7. When $j = 1$ then the possible first terms for the series are $0/3$ and $2/3$. These are the left endpoints of the remaining intervals from the first step of the Cantor construction.

When $j = 2$ then the possible first terms for the series are $0/9$ and $2/9$. When these are added to the two possibilities for the $J = 1$ term we get four possible points. In fact they are the left endpoints of the four remaing intervals. And so forth for $j = 3, 4, \ldots$.

Every element of the Cantor set is the limit of a sequence of the endpoints. This follows from the construction. So the series

$$\sum_{j=1}^{\infty} \mu_j a_j$$

indeed describes the Cantor set. And, as we have already indicated, the finite partial sums describe the endpoints.

9. Using the characterization of the Cantor set given in Exercise 7, we see that

$$C+C = \left\{ \sum_{j=1}^{\infty} \mu_j a_j + \sum_{j=1}^{\infty} \lambda_j a_j : \mu_j = 0 \text{ or } 2, \lambda_j = 0 \text{ or } 2 \right\} = \left\{ \sum_{j=1}^{\infty} \tau_j a_j : \tau_j = 0 \text{ or } 2 \text{ or } 4 \right\} .$$

It is not difficult to see that the expression on the right equals the interval $[0, 2]$.

* **11.** Certainly any sequence of the form $a_j = \rho^j$ for $0 < \rho < 1/3$ will do the job.

4.5 Connected and Disconnected Sets

1. The set $X = [s, t]$ is connected. To see this, suppose to the contrary that there exist open sets U and V such that $U \cap X \neq \emptyset, V \cap X \neq \emptyset, (U \cap X) \cap (V \cap X) = \emptyset$, and

$$X = (U \cap X) \cup (V \cap X) .$$

Choose $a \in U \cap X$ and $b \in V \cap X$. Set

$$\alpha = \sup (U \cap [a, b]\}) .$$

Now $[a, b] \subseteq X$ hence $U \cap [a, b]$ is disjoint from V. Thus $\alpha \leq b$. But ${}^cV$ is closed hence $\alpha \notin V$. It follows that $\alpha < b$.

If $\alpha \in U$ then, because U is open, there exists an $\widetilde{\alpha} \in U$ such that $\alpha < \widetilde{\alpha} < b$. This would mean that we chose α incorrectly. Hence $\alpha \notin U$. But $\alpha \notin U$ and $\alpha \notin V$ means $\alpha \notin X$. On the other hand, α is the supremum of a subset of X (since $a \in X, b \in X$, and X is an interval). Since X is a closed interval, we conclude that $\alpha \in X$. This contradiction shows that X must be connected.

3. Let $S = \{q \in [0, 1] : q \in \mathbb{Q}\}$.

5. If a and b are distinct elements of the Cantor set, then let $\epsilon = |b-a| > 0$. Choose j so large that $3^{-(j-3)} < \epsilon$. Then a and b live in different closed intervals in the set S_j in the construction of the Cantor set. So a and b must be in different connected components of C.

7. Let $A = [0,1]$ and $B = [2,3]$. Then A and B are each connected but $A \cup B$ is not connected.

9. In general $A \cap B$ will be disconnected. Consider $A = [0,1]$ and B equals the Cantor set.

4.6 Perfect Sets

1. Let $U_j = (-\infty, -1 - 1/j) \cup (1 + 1/j, +\infty)$. Then it is clear that $U_1 \subseteq U_2 \subseteq \cdots$ and each U_j has bounded, nonempty complement. But

$$\bigcup_j U_j \not\supseteq [-1/2, 1/2] .$$

Hence $\bigcup_j U_j \neq \mathbb{R}$.

3. If $S \subseteq \mathbb{R}$ is perfect and $T \subseteq \mathbb{R}$ is perfect, then consider $S \times T \subseteq \mathbb{R} \times \mathbb{R}$. Certainly $S \times T$ is closed. If $(s,t) \in S \times T$ then s is the limit of a sequence $s_j \in S$ and t is the limit of a sequence $t_j \in T$ so (s,t) is the limit of the sequence (s_j, t_j). Hence $S \times T$ is perfect.

5. Let $A = [0,1] \cap \mathbb{Q}$ and $B = [0,1] \setminus \mathbb{Q}$. Then $A \cup B = [0,1]$ is perfect, but neither A nor B is perfect.

7. A nontrivial, closed interval $[a,b]$ is perfect and the Cantor set is perfect. The interior of $[a,b]$ is (a,b) while the interior of the Cantor set is $\emptyset$. So it is hard to say anything about the interior of a perfect set.

* **11.** Let S be any closed set and let $C \subseteq S$ be the set of condensation points. Then $S \setminus C$ must be countable or else, by Exercise 10, $S \setminus C$ would have condensation points.

Chapter 5

Limits and Continuity of Functions

5.1 Definition and Basic Properties of the Limit of a Function

1. Say that the limit of f is ℓ_1 and the limit of g is ℓ_2. Let $\epsilon > 0$. Choose $\delta_1 > 0$ so that $0 < |x - c| < \delta_1$ implies that $|f(x) - \ell_1| < \epsilon/2$. Choose $\delta_2 > 0$ so that $0 < |x - c| < \delta_2$ implies that $|g(x) - \ell_2| < \epsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - c| < \delta$, then

$$\ell_1 = (\ell_1 - f(x)) + (f(x) - g(x)) + (g(x) - \ell_2) + \ell_2 < \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} + \ell_2 \,.$$

In summary, we see that, for any $\epsilon > 0$,

$$\ell_1 \le \ell_2 + \epsilon \,.$$

It follows that

$$\ell_1 \le \ell_2 \,.$$

3. Let f be a function with domain a set of the form $(a, c) \cup (c, b)$. We say that $\lim_{x \to c} f(x) = \ell$ if, given any open neighborhood U of ℓ, there is an open neighborhood V of c so that $f(V) \subseteq U$.

5. Define

$$f(x) = \begin{cases} 0 & \text{if} \quad x \text{ is irrational} \\ 1/q & \text{if} \quad x = p/q \text{ is a rational in lowest terms}. \end{cases}$$

Then it is easy to see that $\lim_{x\to c} f(x) = 0$ at each point c, but f is discontinuous at each rational number.

7. We shall prove part **(a)**. Part **(c)** is similar. Let $\epsilon > 0$. Choose $\delta_1 > 0$ so that $0 < |x - P| < \delta_1$ implies that $|f(x) - \ell| < \epsilon/2$. Choose $\delta_2 > 0$ so that $0 < |x - P| < \delta_2$ implies that $|g(x) - m| < \epsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$. If $|x - P| < \delta$ then

$$|(f(x) + g(x)) - (\ell + m)| \le |f(x) - \ell| + |g(x) - m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

Hence

$$\lim_{x\to P}(f(x) + g(x)) = \ell + m\,.$$

9. Let $\epsilon > 0$. Set $\delta = \epsilon$. If $0 < |x - 0| < \delta$ then

$$|f(x) - 0| = |x\sin(1/x) - 0| = |x\sin x| \le |x| < \delta = \epsilon\,.$$

Thus

$$\lim_{x\to 0} f(x) = 0 = f(0)\,.$$

We see that f is continuous at the origin.

Now let $\epsilon = 1/2$. Let $x_j = 1/((j + (1/2))\pi)$ for $j = 1, 2, \ldots$. Then

$$g(x_j) = 1$$

for j even and

$$g(x_j) = -1$$

for j odd. So $\lim_{x\to 0} g(x)$ does not exist.

* **11.** See the solution to Exercise 5 above.

5.2 Continuous Functions

1. This function is discontinuous at every point except the origin. First suppose that c is a rational number unequal to 0. If x_j is a sequence of irrationals approaching c then $f(x_j) = 0$ but $f(c) = c$. So $\lim_{x\to c} f(x) \ne f(c)$. Now suppose that c is irrational. If x_j are rational numbers approaching c then $f(x_j) = x_j$. Hence $\lim_{j\to\infty} f(x_j) =$

$\lim_{j\to\infty} x_j = c$. But $f(c) = 0$. So f is discontinuous at c. Finally suppose that $c = 0$. If x_j is a sequence of irrationals approaching c then $f(x_j) = 0$ while if y_j is a sequence of rationals approaching c then $f(y_j) = y_j \to c = 0$. In any event, $\lim_{x\to 0} f(x) = 0 = f(0)$. So f is continuous at 0.

3. Refer to the solution of Exercise 2 in Section 5.1.

5. Let $f(x, y) = x$.

7. Refer to the solution of Exercise 5 in Section 5.1.

9. For simplicity let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Let E be a closed set in $\mathbb{R}$. Consider $f^{-1}(E)$. If $f^{-1}(E)$ is not closed, then there is a convergent sequence of points $x_j \in f^{-1}(E)$ such that the limit point x_0 is *not* in $f^{-1}(E)$. This means that the $f(x_j)$ lie in E but $f(x_0)$ does not lie in E. Most importantly, the continuity of f implies that $f(x_j) \to f(x_0)$. And E is closed so $f(x_0)$ must lie in E. That is a contradiction.

The reasoning for the converse direction is similar.

* **11.** The composition of uniformly continuous functions is indeed uniformly continuous. To see this, suppose that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and $g : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Let $\epsilon > 0$. Choose $\eta > 0$ so that, if $|s - t| < \eta$ then $|g(s) - g(t)| < \epsilon$. Next choose $\delta > 0$ so that, if $|x - y| < \delta$ then $|f(x) - f(y)| < \eta$.

Now if $|x - y| < \delta$ then $|f(x) - f(y)| < \eta$ and therefore $|g(f(x)) - g(f(y))| < \epsilon$. So $g \circ f$ is uniformly continuous.

5.3 Topological Properties and Continuity

1. Certainly f is nonnegative at all x. But the function $f(x) = (x - \sqrt{2})^2$ is positive for all rational x and equal to zero at $x = \sqrt{2}$.

3. Let $f(x) = x^2$ and $U = (-1, 1)$. Then $f(U) = [0, 1)$, which is not open.

5. Let $f : \mathbb{R} \to \mathbb{R}$. Define $f(x) = x^2$. Then the intervals $[-2, -1]$ and $[1, 2]$ are disjoint, but $f([-2, -1]) = f([1, 2]) = [1, 4]$.

By contrast, if C and D are disjoint then $f^{-1}(C)$ and $f^{-1}(D)$ are disjoint. That is because if $f(x) \in C$ then certainly $f(x) \notin D$ and if $f(x) \in D$ then certainly $f(x) \notin C$.

7. Let $f(x) = \sin x$ and let $A = [0, \pi/2] \cup [2\pi, 5\pi/2] \cup [4\pi, 9\pi/2] \cup \cdots \cup [2k\pi, (4k+1)\pi/2]$. Then A has k connected components, but $f(A) = [0, 1]$, which has one connected component.

9. The function $f : (0, 1) \to (0, 1)$ given by $f(x) = x^2$ has no fixed point.

11. Now $E \cup F$ is closed so the complement of $(E \cup F)$ is open. Therefore

$$^c(E \cup F) = \bigcup_{j=1}^{\infty} I_j$$

is the pairwise disjoint union of open intervals. Write $I_j = (a_j, b_j)$. Now define

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in E \\ 1 & \text{if} \quad x \in F \\ \dfrac{x-a}{b-a} & \text{if} \quad x \in I_j, a_j \in E, b_j \in F \\ \dfrac{-1}{b-a}(x-b) & \text{if} \quad x \in I_j, a_j \in F, b_j \in E \\ \dfrac{4}{(b-a)^2+4}\left[\left(x - \dfrac{a+b}{2}\right)^2 + 1\right] & \text{if} \quad x \in I_j, a_j \in F, b_j \in F\,. \end{cases}$$

Then f is continuous, $E = f^{-1}(0)$, and $F = f^{-1}(1)$.

* **13.** If the conclusion is not true, then there is an $\epsilon > 0$ and points x arbitrarily close to a so that $|f(x) - c| > \epsilon$ and $|f(x) - d| > \epsilon$. But this implies that $f^{-1}([c+\epsilon, d-\epsilon])$ contains points x that are arbitrarily near to a. Thus f^{-1} of the compact set $[c+\epsilon, d-\epsilon]$ is not compact. Contradiction.

5.4 Classifying Discontinuities and Monotonicity

1. Write $A = \{a_j\}_{j=1}^{\infty}$. Define

$$f(x) = \begin{cases} 0 & \text{if} \quad x < a_1 \\ j & \text{if} \quad a_j \le x < a_{j+1} \end{cases}$$

Then it is clear that f is increasing and has a jump discontinuity at each a_j.

This is impossible for an uncountable set A because the only possible discontinuities for an increasing function are jump discontinuities. And there can only be countably many jump discontinuities.

3. Let

$$f(x) = \begin{cases} -1 & \text{if} \quad x \le 0 \\ 1 & \text{if} \quad x > 0 \,. \end{cases}$$

Then f is discontinuous at the origin, but $f^2(x) \equiv 1$ is continuous at every point.

If f^3 is continuous at every point, then

$$f(x) = [f^3(x)]^{1/3}$$

is the composition of continuous functions, so is certainly continuous.

5. The set $[a, b]$ is connected, so $f([a, b])$ will be connected. And the only connected sets are intervals, so $f([a, b])$ is an interval. As we indicated in Exercise 2 of Section 5.2, $f([a, b])$ can be an interval of any of the four types.

* **9.** Now let x, y lie in the domain of f. Assume without loss of generality that $f(x) = 0$ and we shall prove that f is continuous at x. For $0 \le t \le 1$, we know that

$$f((1-t)x + ty) \le (1-t)f(x) + tf(y) \,. \qquad (*)$$

We think of t as positive and small, so that $(1-t)x + ty$ is close to x. Then we see that

$$|f((1-t)x + ty)| \le t|f(y)| \,.$$

That shows that

$$\lim_{s\to x} f(s) = 0 = f(x)\,,$$

so that f is continuous at x.

In calculus we learn that a twice differentiable function is "concave up" if $f''(x) \geq 0$. Now we look at

$$\begin{aligned}(1-t)f(a)+tf(b)-f((1-t)a+tb) &= t[f(b)-f((1-t)a+tb)] \\ &\qquad -(1-t)[f((1-t)a+tb)-f(a)] \\ &= tf'(\xi)(1-t)(b-a) \\ &\qquad -(1-t)f'(\widetilde{\xi})t(b-a)\,,\end{aligned}$$

where we have applied the Mean Value Theorem twice. Now we shall apply the Mean Value Theorem again to rewrite the last line as

$$t(1-t)(b-a)f''(\widetilde{\widetilde{\xi}}) \geq 0\,.$$

We have proved, assuming that $f'' \geq 0$, that

$$(1-t)f(a)+tf(b)-f((1-t)a+tb) \geq 0\,,$$

which is the same as

$$f((1-t)a+tb) \leq (1-t)f(a)+tf(b)\,.$$

So an f with nonnegative second derivative is convex according to our new definition.

11. The assertion is true, and remains true if the function is continuously differentiable.

Chapter 6

Differentiation of Functions

6.1 The Concept of Derivative

1. This assertion is false for every $k \geq 2$. We give just a few examples:

- Let $f(x) = |x|$. Then $f^2(x) = x^3$ is differentiable at all points, but f fails to be differentiable at the origin.
- Let $f(x) = x^{1/3}$. Then $f^3(x) = x$ is differentiable at all points, but f fails to be differentiable at the origin.
- Let

  $$f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x & \text{if} \quad x \geq 0 \,. \end{cases}$$

  Then

  $$f^4(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x^4 & \text{if} \quad x \geq 0 \,. \end{cases}$$

  We see that f^4 is certainly differentiable at the origin but f is not.

3. This function is certainly not differentiable at any rational. If $q \in \mathbb{R}$ is rational and positive, then write $q = a/b$, where a and b are positive integers and the fraction is in lowest terms. Now we examine the Newton quotient

$$\mathcal{Q} = \frac{f(q+h) - f(q)}{h} \,.$$

If $q+h$ is irrational (say h is a small multiple of $\sqrt{2}$) then $\mathcal{Q} = (-1/b)/h$ blows up as $h \to 0$. So f is not differentiable at q.

The analysis of differentiability at irrational points is more difficult and we omit it.

5. Let f be defined on the interval $(a, b]$. We say that f is *left continuous* at the point b if

$$\lim_{h\to 0^-} f(b+h) = f(b)\,.$$

We say that f is *left differentiable* at the point b if

$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}$$

exists. We denote the left derivative of f by f'_ℓ. Right continuity and right differentiability are defined similarly.

Now suppose that f is left differentiable at b as above. Then

$$\begin{aligned} \lim_{h\to 0^-} [f(b+h)-f(b)] &= \lim_{h\to 0^-} \left(\frac{f(b+h)-f(b)}{h} \cdot h \right) \\ &= \left(\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} \right) \cdot (\lim_{h\to 0^-} h) \\ &= f'_\ell(b) \cdot 0 = 0\,. \end{aligned}$$

Thus f is left continuous at b.

The result for right differentiability and right continuity is proved in just the same fashion.

7. We see that the discontinuity is of the second kind.

* **13.** Let f be a function that is differentiable at a point x in the domain interval I for f.

(a) We can calculate that

$$\lim_{h\to 0} \frac{f^2(x+h)-f^2(x)}{h} = \lim_{h\to 0} \left(\left[\frac{f(x+h)-f(x)}{h} \right] \cdot [f(x+h)+f(x)] \right), . = \lim_{h\to 0} \left[\frac{f(\right.$$

Since f is differentiable at x, the first expression in brackets tends to $f'(x)$. Since f is then continuous at x, the second expression in brackets tends to $2f(x)$. Altogether then,

$$(f^2)'(x) = 2f(x) \cdot f'(x)\,.$$

(b) Applying the result of part **(a)** to the function $f+g$, we find that

$$\left[(f+g)^2\right]'(x) = 2(f+g)(x) \cdot (f+g)'(x) .$$

(c) We can write this out as

$$[f^2]'(x)+[2fg]'(x)+[g^2]'(x) = 2f(x)\cdot f'(x)+2f(x)\cdot g'(x)+2f'(x)\cdot g(x)+2g(x)\cdot g'(x) .$$

Cancelling out matching terms (using part **(a)**), we find that

$$[2fg]'(x) = 2f(x) \cdot g'(x) + 2f'(x) \cdot g(x)$$

or

$$[fg]'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x) .$$

6.2 The Mean Value Theorem and Applications

1. We can calculate that

$$|f(x)| = \left|\int_0^x f'(t)\,dt\right| \le \int_0^x |f'(t)|\,dt \le \int_0^x |f(t)|\,dt .$$

Let

$$F(x) = \int_0^x |f(t)|\,dt .$$

Then F is an antiderivate of $|f|$. And we have

$$|f(x)| \le F(x) .$$

This can be rewritten as

$$\frac{|f(x)|}{F(x)} \le 1 .$$

Integrating both sides from 0 to x gives

$$\log F(x) \le x .$$

This can be rewritten as

$$F(x) \le e^x .$$

From this we conclude that

$$|f(x)| \le F(x) \le e^x .$$

3. If f is neither increasing or decreasing on I, then there are two possibilities: **(i)** $f' \equiv 0$, in which case f is constant, or **(ii)** f is increasing on some subinterval J of I and f is decreasing on some other subinterval J' of I. In the second scenario, it must be that f has an interior local minimum P or an interior local maximum Q. That would be a contradiction.

5. We may write

$$f'(x) = f'(0) + \int_0^x f''(t)\,dt \geq f'(0) + \int_0^x c\,dt \geq f'(0) + cx\,.$$

Therefore

$$f(x) = f(0)+\int_0^x f'(t)\,dt \geq f(0)+\int_0^x (f'(0)+ct)\,dt = f(0)+f'(0)x+\frac{cx^2}{2}\,.$$

Clearly the function on the right is not bounded above, so f is not bounded above.

7. The derivative of f^k is $k \cdot f^{k-1} \cdot f'$. So clearly we need $f > 0$ for the desired conclusion to be true.

9. Let $\varphi(t) = t^{1/2}$. We asked to evaluate the behavior at $+\infty$ of $\varphi(x+1) - \varphi(x)$. By the Mean Value Theorem we have

$$\varphi(x+1) - \varphi(x) = [(x+1) - x] \cdot \varphi'(\xi)$$

for some ξ between x and $x+1$. Thus

$$|\varphi(x+1) - \varphi(x)| \leq 1 \cdot \frac{1}{2}x^{-1/2}\,.$$

As $x \to +\infty$, the righthand side clearly tends to 0. So the expression $\varphi(x+1) - \varphi(x)$ tends to 0.

11. Applying the usual Mean Value Theorem to h we see that

$$h(b) - h(a) = (b-a) \cdot h'(\xi) \qquad (*)$$

for some ξ between a and b. Writing out $(*)$ gives

$$\begin{aligned} &[g(b)(f(b) - f(a)) - f(b)(g(b) - g(a))] \\ &\qquad - [g(a)(f(b) - f(a)) - f(a)(g(b) - g(a))] \\ &= (b-a)\,[g'(\xi)(f(b) - f(a)) - f'(\xi)(g(b) - g(a))]\,. \end{aligned}$$

Dividing through by $g(b) - g(a)$, we obtain

$$\begin{aligned} g(b) \cdot \frac{f(b)-f(a)}{g(b)-g(a)} - f(b) & \\ -g(a) \cdot \frac{f(b)-f(a)}{g(b)-g(a)} + f(a) & \\ = \quad (b-a)g'(\xi)\frac{f(b)-f(a)}{g(b)-g(a)} - f'(\xi)(b-a)\,. & \end{aligned}$$

With some elementary algebra, this simplifies to

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}\,.$$

6.3 More on the Theory of Differentiation

1. We write

$$\begin{aligned} \left|\frac{f(x+h)+f(x-h)-2f(x)}{h^2}\right| &= \left|\frac{(f(x+h)-f(x))-(f(x)-f(x-h))}{h^2}\right| \\ &= \left|\frac{f'(\xi)\cdot h - f'(\widehat{\xi})\cdot h}{h^2}\right| \\ &= \left|\frac{f'(\xi) - f'(\widehat{\xi})}{h}\right| \\ &= \left|\frac{f''(\widehat{\widehat{\xi}})\cdot h}{h}\right| \\ &= \left|f''(\widehat{\widehat{\xi}})\right| \\ &\leq C\,. \end{aligned}$$

3. When ℓ is even then $|x|^\ell = x^\ell$, which is infinitely differentiable. So this case is not interesting.

The case $\ell = 3$ is typical of the odd case. Notice that

$$f(x) = |x|^3 = \begin{cases} -x^3 & \text{if} \quad x \leq 0 \\ x^3 & \text{if} \quad x > 0\,. \end{cases}$$

Then

$$f'(x) = \begin{cases} -3x^2 & \text{if} \quad x \le 0 \\ 3x^2 & \text{if} \quad x > 0 \end{cases}$$

and

$$f''(x) = \begin{cases} -6x & \text{if} \quad x \le 0 \\ 6x & \text{if} \quad x > 0 \,. \end{cases}$$

We see that f'' is not differentiable at the origin. But in fact f'' is Lipschitz-1. So $f \in C^{2,1}$.

In general, when ℓ is odd, $f(x) = |x|^\ell$ is in $C^{(\ell-1,1)}$.

5. If f is not invertible then f is not one-to-one. So there exist $c < d$ such that $f(c) = f(d)$. It follows then that there is a point ξ, $c < \xi < d$, which is either a local maximum or a local minimum for f. But then Fermat's lemma tells us that $f'(\xi) = 0$. Contradiction.

Now consider

$$\frac{f^{-1}(x+h) - f^{-1}(x)}{h} \,. \tag{$*$}$$

We write $x = f(y)$ and $x + h = f(\widetilde{y})$. Then we may rewrite $(*)$ as

$$\frac{\widetilde{y} - y}{f(\widetilde{y}) - f(y)} = \frac{1}{[f(\widetilde{y}) - f(y)]/[\widetilde{y} - y]} \,. \tag{$**$}$$

If $h \to 0$ then $x + h \to x$. Since f is invertible, we may then conclude that $\widetilde{y} \to y$. So equation $(**)$ converges to $1/f'(y)$. So f^{-1} is differentiable.

We may derive a formula for $(f^{-1})'$ as follows. We know that

$$f \circ f^{-1}(x) = x \,.$$

Differentiating both sides gives

$$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$$

or

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \,.$$

This is consistent with the formula that we derived in the last paragraph.

7. We know from Exercise 1 above and others like it that, if f is twice differentiable at x then

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x) \, .$$

However, as Exercise 21 of the last section shows, the converse is not true. There is not a simple characterization of the second derivative that is analogous to the Newton quotient.

* **9.** The function $f(x) = x \cdot \ln|x|$ is actually in Lip-α for any $\alpha < 1$. One needs only check this assertion at the origin, and for that refer to Exercise 4 of Section 6.2 above.

The function $g(x) = |x|/\ln|x|$ is in $C^{0,1}$ as a simple calculation will show.

Chapter 7

The Integral

7.1 Partitions and the Concept of Integral

1. If f is not bounded then there are points $s_j \in [a, b]$ so the $f(s_j) \to \infty$. For simplicity let us assume that the values converge to $+\infty$. Let $N > 0$ and choose J so large that $j > J$ implies that $f(s_j) > N$. Set up a partition $\mathcal{P}$ given by $x_0 < x_1 < \cdots < x_k$ so that $s_{N+1} \in I_1$, $s_{N+2} \in I_2$, $\ldots$, $s_{N+k} \in I_k$. Now consider the Riemann sum

$$\mathcal{R}(f, \mathcal{P}) = \sum_j f(s_j)\Delta_j \geq N \cdot (b - a) \,.$$

This shows that the value of the integral is at least $N \cdot (b-a)$. Since $N > 0$ was arbitrary, we conclude that the integral does not exist.

3. The function f is continuous so it is certainly Riemann integrable.

7. The function f is, in effect, continous on each of finitely many subintervals of $[a, b]$. One simply applies Theorem 7.10 on each of these subintervals.

9. We shall follow the scheme presented in Remark 7.7. Given $\epsilon > 0$, choose $\delta > 0$ as in the definition of the integral. Fix a partition $\mathcal{P}$ with mesh smaller than δ. Let $K + 1$ be the number of points in $\mathcal{P}$. Choose points $t_j \in I_j$ so that $|f(t_j) - \sup_{I_j} f| < \epsilon/(2(K + 1))$; also choose points $t'_j \in I_j$ so that $|f(t'_j) - \inf_{I_j} f| < \epsilon/(2(K + 1))$. By applying the definition of the integral to this choice of t_j and t'_j we find that

$$\sum_j \left(\sup_{I_j} f - \inf_{I_j} f \right) \Delta_j < 2\epsilon\,.$$

The result follows from Remark 7.7.

7.2 Properties of the Riemann Integral

1. Let f be continuous and non-negative on $[0,1]$. Let $M = \sup_{[0,1]} f$. Let $\epsilon > 0$ be fixed. Since f is continuous, there exists a $\delta > 0$ and an interval I_δ of length $\delta, I_\delta \subseteq [0,1]$, such that for all $x \in I_\delta$,

$$M - \epsilon \leq f(x).$$

Then

$$\begin{aligned} \left[\int_0^1 f(t)^n dt \right]^{\frac{1}{n}} &\geq \left[\int_{I_\delta} f(t)^n dt \right]^{\frac{1}{n}} \\ &\geq [(M-\epsilon)^n \delta]^{\frac{1}{n}} \\ &= (M-\epsilon)\delta^{\frac{1}{n}}. \end{aligned}$$

Thus,

$$\begin{aligned} \liminf_{n\to\infty} \left[\int_0^1 f(t)^n dt \right]^{\frac{1}{n}} &\geq \lim_{n\to\infty} (M-\epsilon)\delta^{\frac{1}{n}} \\ &= M - \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary we have

$$\liminf_{n\to\infty} \left[\int_0^1 f(t)^n dt \right]^{\frac{1}{n}} \geq M.$$

Since

$$\limsup_{n\to\infty} \left[\int_0^1 f(t)^n dt \right]^{\frac{1}{n}} \leq M$$

is trivial, we are done.

3. If f is integrable on $[a,b]$, then

$$\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} f(x)\,dx = \int_a^b f(x)\,dx\,.$$

So no new integrable functions arise in this case.

Now let $g(x) = x^{-1/2}$ on the interval $(0,1)$. We see that

$$\lim_{\epsilon \to 0^+} \int_{\epsilon}^{1} x^{-1/2}\,dx = \lim_{\epsilon \to 0^+} 2 \cdot x^{1/2}\Big|_{\epsilon}^{1} = \lim_{\epsilon \to 0^+} 2 - 2\epsilon^{1/2} = 2\,.$$

So g is an example of an unbounded function that can be integrated by this new definition.

Define

$$h(x) = \begin{cases} 6 & \text{if} \quad 1/3 < x \le 1/2 \\ -6 & \text{if} \quad 1/4 < x \le 1/3 \\ 20/3 & \text{if} \quad 1/5 < x \le 1/4 \\ -15/2 & \text{if} \quad 1/6 < x \le 1/5 \end{cases}$$

and so forth. You can see that the graph of this function is alternating positive and negative bumps, and the area of the jth bump is $1/j$. Thus it is straightforward to see that the limit

$$\lim_{\epsilon \to 0^+} \int_{\epsilon}^{1/2} h(x)\,dx$$

exists.

Of course

$$|h|(x) = \begin{cases} 6 & \text{if} \quad 1/3 < x \le 1/2 \\ 6 & \text{if} \quad 1/4 < x \le 1/3 \\ 20/3 & \text{if} \quad 1/5 < x \le 1/4 \\ 15/2 & \text{if} \quad 1/6 < x \le 1/5 \end{cases}$$

The graph of $|h|$ is a sequence of bumps of area $1/j$, but now they are all positive. Recall that the series $\sum_j (-1)^j/j$ converges, but the harmonic series $\sum_j 1/j$ diverges. This is why h is integrable by our new methodology but $|h|$ is not.

7. Let $\epsilon > 0$. Choose $\delta > 0$ so that, if $\mathcal{P}$ is a partition of $[a, b]$ with mesh less than δ, then

$$|\mathcal{R}(f, \mathcal{P}) - \int_a^b f(x)\, dx| < \epsilon/(\alpha + 1)\,.$$

Then

$$|\mathcal{R}(\alpha f, \mathcal{P}) - \int_a^b \alpha f(x)\, dx| < \alpha \cdot \epsilon/(\alpha + 1) < \epsilon\,.$$

Therefore

$$|\mathcal{R}(\alpha f, \mathcal{P}) - \alpha \cdot \int_a^b f(x)\, dx| < \epsilon\,.$$

That proves the result.

* **9.** First consider

$$\int_\eta^1 \frac{\cos 2r - \cos r}{r} dr.$$

We write this as

$$\begin{aligned} \int_\eta^1 \frac{\cos 2r}{r} dr - \int_\eta^1 \frac{\cos r}{r} dr &= \int_{2\eta}^2 \frac{\cos s}{s} ds - \int_\eta^1 \frac{\cos s}{s} ds \\ &= -\int_\eta^{2\eta} \frac{\cos s}{s} ds + \int_1^2 \frac{\cos s}{s} ds. \end{aligned}$$

The second integral obviously exists and is finite. Let $\epsilon > 0$. If $\eta > 0$ is small enough then $1 - \epsilon < \cos s < 1$ on $[\eta, 2\eta]$. Thus

$$(1 - \epsilon) \log 2 = \int_\eta^{2\eta} \frac{1 - \epsilon}{s} dx < \int_\eta^{2\eta} \frac{coss}{s} dx \leq \int_\eta^{2\eta} \frac{1}{s} ds = \log 2.$$

We conclude that

$$\lim_{\eta \to 0} \int_\eta^{2\eta} \frac{\cos s}{s} ds$$

exists. Therefore

$$\lim_{\eta \to 0} \int_{\eta}^{1} \frac{\cos s}{s} ds$$

exists.

The integral

$$\lim_{\eta \to 0} \int_{1}^{1/\eta} \frac{\cos s}{s} ds$$

is treated similarly.

7.3 Change of Variable and Related Ideas

1. We will concentrate on the one-sided derivative of F at a. Fix $\epsilon > 0$. Choose a $\delta > 0$ so that, for $x > a$, $|x-a| < \delta$, we have $|f(x)-f(a)| < \epsilon$. For $\delta > h > 0$ we calculate that

$$\begin{aligned}
\left|\frac{F(a+h)-F(a)}{h} - f(a)\right| &= \left|\frac{\int_a^{a+h} f(t)\,dt - \int_a^a f(t)\,dt}{h} - f(a)\right| \\
&= \left|\frac{\int_a^{a+h} f(t)\,dt}{h} - f(a)\right| \\
&= \left|\frac{\int_a^{a+h} f(t)\,dt - \int_a^{a+h} f(a)\,dt}{h}\right| \\
&= \left|\frac{\int_a^{a+h} f(t) - f(a)\,dt}{h}\right| \\
&\leq \frac{\int_a^{a+h} |f(t) - f(a)|\,dt}{h} \\
&\leq \frac{\int_a^{a+h} \epsilon\,dt}{h} \\
&= \epsilon\,.
\end{aligned}$$

This shows that F is right differentiable at a and the derivative is equal to $f(a)$.

5. Since f is Riemann integrable, f is bounded by some number M. Let $\epsilon > 0$ and choose $\delta > 0$ so that if the mesh of partitions $\mathcal{P}$ and $\mathcal{P}'$ is less than δ and if $\mathcal{Q}$ is the common refinement of $\mathcal{P}$ and $\mathcal{P}'$, then

$$|\mathcal{R}(f,\mathcal{P}) - \mathcal{R}(f,\mathcal{Q})| < \epsilon/(2M)\,. \tag{$\star$}$$

Let I_j be the intervals that arise in the partition $\mathcal{P}$ and let $\widetilde{I}_\ell$ be the intervals that arise in the partition $\mathcal{Q}$. Note that each $\widetilde{I}_\ell$ is contained in some interval I_j. Then it follows that

$$\left|\mathcal{R}(f^2,\mathcal{P}) - \mathcal{R}(f^2,\mathcal{Q})\right| = \left|\sum_{I_j}\sum_{\widetilde{I}_\ell\subseteq I_j} f^2(s_j)\widetilde{\Delta}_\ell - \sum_{I_j}\sum_{\widetilde{I}_\ell\subseteq I_j} f^2(\widetilde{s}_\ell)\widetilde{\Delta}_\ell\right|\,. \tag{$*$}$$

But

$$|f^2(s_j) - f^2(\widetilde{s}_\ell)| = |(f(s_j) + f(\widetilde{s}_\ell)\cdot f(s_j) - f(\widetilde{s}_\ell)| \le 2M\cdot|f(s_j) - f(\widetilde{s}_\ell)|\,.$$

So $(*)$ can be estimated by

$$\sum_{I_j}\sum_{\widetilde{I}_\ell\subseteq I_j} 2M\cdot|f(s_j) - f(\widetilde{s}_\ell)|\widetilde{\Delta}_\ell\,.$$

And we know from equation $(\star)$ that the righthand side is less than ϵ. It follows that f^2 is Riemann integrable.

7. Some reasons are

- Differentiation comes from taking a difference while integration comes from taking a summation.
- Differentiation measures rate of change while integration measures aggregation.
- Differentiation reduces the complexity of a polynomial while integration increases the complexity.

9. A Riemann sum for $\int_0^1 x^2\,dx$, with equally spaced partition having k intervals, is

$$\sum_{j=1}^k \left(\frac{j}{k}\right)^2 \frac{1}{k} = \frac{1}{k^3}\sum_{j=1}^k j^2 = \frac{1}{k^3}\cdot\frac{2k^3+3k^2+k}{6}\,.$$

In the last equality we have used a result from Exercise 1 of Section 2.4. Now this last expression tends to 1/3 as $k \to \infty$. And it is easy to calculate, using the Fundamental Theorem of Calculus, that $\int_0^1 x^2\,dx = 1/3$.

7.4 Another Look at the Integral

1. We observe that

$$\int_2^6 t^2 d\beta(t) = \int_2^6 t^2\,dt + \int_2^6 t^2\,d[t]\,,$$

where $[t]$ is the greatest integer function. The first integral on the right is

$$\int_2^6 t^2\,dt = \frac{208}{3}$$

while the second integral equals

$$\int_2^6 t^2\,d[t] = 3^2 + 4^2 + 5^2 + 6^2 = 86\,.$$

In sum,

$$\int_2^6 t^2\,dt = \frac{208}{3} + 86 = \frac{466}{3}\,.$$

3. Suppose that $p(x) = a_0 + a_1 x + \cdots + a_k x^k$ is a polynomial. For each j, take $\alpha(x) = \alpha_j(x) = x^j$. Then our hypothesis is that

$$0 = \int_a^b p(x)\,d\alpha_j(x) = \int_a^b p(x) j x^{j-1}\,dx\,.$$

The equations

$$0 = \int_a^b p(x)x^j = 0$$

for $j = 0, 1, 2, \ldots, (k+1)$ imply, with a little algebra, that $a_0 = a_1 = \cdots = a_k = 0$. So p is the zero polynomial.

5. We calculate that

$$\int_0^3 x^2\,d\alpha(x) = 1^2 + 2^2 + 3^2 = 14\,.$$

7. We shall prove part **(a)**. Simply observe that

$$|\mathcal{U}(f+g,\mathcal{P},\alpha)-\mathcal{L}(f+g,\mathcal{P},\alpha)| \leq |\mathcal{U}(f,\mathcal{P},\alpha)-\mathcal{L}(f,\mathcal{P},\alpha)|+|\mathcal{U}(g,\mathcal{P},\alpha)-\mathcal{L}(g,\mathcal{P},\alpha)| .$$

Now, given any $\epsilon > 0$, we may choose a partition $\mathcal{P}$ so that

$$|\mathcal{U}(f,\mathcal{P},\alpha) - \mathcal{L}(f,\mathcal{P},\alpha)| < \frac{\epsilon}{2}$$

and a partition $\mathcal{P}'$ so that

$$|\mathcal{U}(g,\mathcal{P},\alpha) - \mathcal{L}(g,\mathcal{P},\alpha)| < \frac{\epsilon}{2} .$$

Then the partition $\mathcal{Q}$ which is the common refinement of $\mathcal{P}$ and $\mathcal{P}'$ will satisfy

$$|\mathcal{U}(f+g,\mathcal{Q},\alpha) - \mathcal{L}(f+g,\mathcal{Q},\alpha)| < \epsilon .$$

9. We calculate that

$$\int_0^\pi f\,d\alpha = \int_0^\pi x^2\,dx^3 = \int_0^\pi x^2 3x^2\,dx = 3\int_0^\pi x^4\,dx = \frac{3\pi^5}{5} .$$

11. The Riemann-Stieltjes integral $\int_0^1 x\,dx$ cannot be represented as a series.

7.5 Advanced Results on Integration Theory

1. Let $f(x) = \sin x/x$. Since $\lim_{x\to 0} f(x) = 1$, we define $f(0) = 1$. Now write

$$\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx .$$

Since f is continuous on the interval $[0,1]$, the first integral on the right certainly exists. For the second, consider

$$\int_1^N \frac{\sin x}{x}\,dx$$

for some $N > 1$. Integrating by parts, we find that

$$\int_1^N \frac{\sin x}{x}\,dx = \frac{-\cos x}{x}\Big|_1^N - \int_1^N \frac{\cos x}{x^2}\,dx .$$

The limit of the first term on the right as $N \to \infty$ clearly exists. Since $|\cos x/x^2| \le 1/x^2$, the limit of the second term on the right also exists. So the full integral exists.

3. The function

$$f(x) = \begin{cases} x\sin(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is continuous on the entire real line. But, because the harmonic series diverges, the sum

$$\sum_{j=1}^{k} \left| f\left(\frac{1}{(2j+1)\pi/2}\right) - f\left(\frac{1}{(2j-1)\pi/2}\right) \right|$$

can be arbitrarily large. So f is not of bounded variation on any interval that has the origin in its interior.

5. In case φ is linear, then

$$\varphi\left(\int_0^1 f(x)\,dx\right) = \int_0^1 \varphi(f(x))\,dx\,.$$

But any convex function is the supremum of linear functions (see [KRA8]). Passing to the supremum gives the desired inequality.

7. Call the interval $[a,b]$. We know from Exercise 2 above that

$$Vf = \int_a^b |f'(x)|\,dx\,.$$

Since f is continuously differentiable, f' is a continuous function on $[a,b]$. So Vf exists and is finite.

9. We have

$$\begin{aligned} \int_0^\pi f\,d\alpha &= \int_0^\pi x^2\,d\sin x = \int_0^\pi x^2 \cos x\,dx \\ &= x^2 \sin x + 2x\cos x - 2\sin x \Big|_0^\pi \\ &= (\pi^2 \cdot 0 + 2\pi \cdot (-1) - 2\sin\pi) - 0 = -2\pi\,. \end{aligned}$$

Chapter 8

Sequences and Series of Functions

8.1 Partial Sums and Pointwise Convergence

1. If the f_j and the limit function f are all bounded from zero by a constant c, then the result is true. This is because then

$$\left|\frac{1}{f_j} - \frac{1}{f}\right| = \left|\frac{f - f_j}{f_j \cdot f}\right| \le \frac{|f_j - f|}{c^2} .$$

3. Let

$$P_N(x) = \sum_{j=0}^{N} (-1)^j \frac{x^{4j+2}}{(2j+1)!} .$$

Of course these are the partial sums of the Taylor series for $\sin x^2$. These converge uniformly on compact intervals to $\sin x^2$.

5. For an integer N that is greater than 2 define the points

$$p_0 = (0,0) \ , \ p_1 = (1/N, 1/N^2) \ , \ p_2 = (2/N, 4/N^2) \ , \ldots ,$$

$$p_{N-1} = ((N-1)/N, (N-1)^2/N^2 \ , \ p_N = (1,1) .$$

Now connect p_0 to p_1 and p_1 to p_2 and so on to connecting p_{N-1} to p_N. This gives the graph of a piecewise linear function L_N with the property that

$$|f(x) - L_N(x)| \le \frac{2}{N} .$$

Finally, set

$$P_1 = L_1 \ , \ P_2 = L_2 - L_1 \ , \ P_3 = L_3 - L_2 \ , \ \ldots \ .$$

Then the series

$$\sum_j P_j$$

converges to f.

7. Consider the series

$$\sum_{j=1}^{\infty} \frac{x^{2j-1}}{(2j-1)!} \ .$$

This is the Taylor series for $f(x) = \sin x$. We can use the Ratio Test to check that the series converges uniformly on compact sets.

9. If the Taylor series $\sum_j a_j x^j$ converges at a point $c \neq 0$ then, for $0 < b < |c|$ and $x \in [-b, b]$, we see that

$$\sum_j \left| a_j x^j \right| = \sum_j a_j \left(\frac{x}{c} \right)^j c^j \ .$$

Now the terms $a_j c^j$ are bounded by some number M. So the jth summand of the series is majorized by

$$M|x/c|^j \ ,$$

and this gives a convergent geometric series. So the Taylor series converges absolutely and uniformly on $[-b, b]$.

11. Notice that

$$\lim_{x \to 0^+} f_j(x) = \lim_{x \to 0^+} x^2/j = 0 = f_j(0) \ .$$

Hence f_j is continuous at the origin. It is obviously continuous elsewhere.

If $x \leq 0$ then $f_j(x) = 0$. If $x > 0$, then

$$|f_j(x)| = \frac{x^2}{j} \to 0$$

as $j \to \infty$. So the f_j converge pointwise to the identically 0 function. However, no matter how large j, $f_j(j^2) = j$. So the convergence is not uniform.

13. Let $0 < \epsilon < 1$. By the Cauchy condition for series, if j is large enough (say $j > J$), then $|g_j(x)| < \epsilon$ for all x. Then

$$\sum_{j=J}^{\infty} |f_j(x) \cdot g_j(x)| \le \sum_{j=J}^{\infty} |f_j(x)| \, .$$

So the series of products converges absolutely and uniformly.

8.2 More on Uniform Convergence

1. The hypothesis means that the sequence of partial sums converges uniformly. But each partial sum is a continuous function. Therefore the limit function is continuous.

3. If ϕ satisfies a Lipschitz-1 condition then things will work out nicely. To illustrate, assume that

$$|\phi(a) - \phi(b)| \le C \cdot |a - b|$$

for all real numbers a and b. Let $\{f_j\}$ be a uniformly convergent sequence of functions with limit function f. Let $\epsilon > 0$. Choose J so large that $j > J$ implies that $|f_j(x) - f(x)| < \epsilon/C$ for all x. Then, for such j,

$$|\phi \circ f_j(x) - \phi \circ f(x)| \le C|f_j(x) - f(x)| < C \cdot \frac{\epsilon}{C} = \epsilon \, .$$

So the $\phi \circ f_j$ converge uniformly.

5. Let

$$f_j(x) = 1 - x^j$$

for $j = 1, 2, \ldots$. Then plainly

$$f_1(x) \le f_2(x) \le \cdots \, .$$

Finally,

$$\lim_{j \to \infty} f_j(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ 0 & \text{if} \quad x = 1 \, . \end{cases}$$

So the limit function is discontinuous.

* **11.** Let $\{q_j\}$ be an enumeration of the rationals. For each positive integer j, define a function φ_j with these properties

- $\varphi_j(x) = 0$ for $x \leq q_j - 10^{-j}$ or $x \geq q_j + 10^{-j}$.
- $\varphi_j(q_j) = j$.
- φ_j is piecewise linear.

Define

$$F_k(x) = \sum_{m=1}^{k} k \cdot \varphi_m(x)\,.$$

Then it is clear that $F_k(q_j) \to \infty$ as $k \to \infty$ for each fixed q_j. It is also clear that

$$\{x : \text{some } \ \varphi_j(x) \text{ is not } 0\}$$

has length

$$2 \cdot 10^{-1} + 2 \cdot 10^{-2} + \cdots = \frac{2}{9}\,.$$

So, for the complementary set of real numbers, the F_k will remain bounded. This will include uncountably many irrationals.

8.3 Series of Functions

1. Let $\epsilon > 0$. For each $x \in [0,1]$, choose j_x large enough that

$$|f(x) - f_{j_x}(x)| < \epsilon\,. \qquad (*)$$

In fact, by the continuity of f and f_{j_x}, the inequality $(*)$ will persist on an interval $I_x = (x - \delta_x, x + \delta_x)$. The intervals I_x cover the compact interval $[0,1]$. So there is a finite subcover

$$I_{x_1}, I_{x_2}, \ldots, I_{x_k}\,.$$

Let $J = \max\{j_{x_1}, j_{x_2}, \ldots, j_{x_k}\}$. It follows now that, if $j > J$ and $x \in [0,1]$, then $x \in I_{j_\ell}$ for some ℓ so that

$$|f(x) - f_j(x)| \leq |f(x) - f_{j_\ell}(x)| < \epsilon\,.$$

Therefore the f_j converge to f uniformly.

3. We know that

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} .$$

Therefore, for any j,

$$e^x \geq c_j \cdot x^j . \qquad (\star)$$

If $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_k x^k$, then pick j much larger than k. It follows then from $(\star)$ that, for N sufficiently large and $x > N$,

$$e^x > |p(x)| .$$

5. Refer to Proposition 3.30. Let $a_j = \sin j$ and $b_j = 1/j$. We calculate that

$$\sin(1/2 + j) = \sin(1/2)\cos j + \cos(1/2)\sin j$$

and

$$\sin(1/2 - j) = \sin(1/2)\cos j - \cos(1/2)\sin j .$$

It follows that

$$\sin(1/2 + j) - \sin(1/2 - j) = 2\cos(1/2)\sin j$$

hence

$$\sin j = \frac{\sin(1/2 + j) - \sin(1/2 - j)}{2\cos(1/2)} .$$

As a result,

$$A_N = \sum_{j=1}^{N} \frac{\sin(1/2 + j) - \sin(1/2 - j)}{2\cos(1/2)} = \frac{\sin(1/2 + N) - \sin(1/2)}{2\cos(1/2)} .$$

We conclude that

$$|A_N| \leq \frac{1}{\cos(1/2)} ,$$

an estimate which is independent of N.

We also observe that $b_1 \geq b_2 \geq \cdots$ and $b_j \to 0$. As a result, Abel's Convergence Test (Theorem 3.31) applies and the series

$$\sum_j \frac{\sin j}{j}$$

converges.

7. Let $\epsilon > 0$. There is a $J > 0$ so that, if $k > j > J$, then

$$\sum_{\ell=j}^{k} g_\ell(x) < \epsilon \text{ for all } x\,.$$

But then it follows that

$$\sum \ell = j^k f_\ell < \epsilon \text{ for all } x\,.$$

So the series $\sum_\ell f_\ell$ converges uniformly.

9. The partial sums are continuous functions, and they converge uniformly. Therefore f is continuous.

8.4 The Weierstrass Approximation Theorem

1. First suppose that f is a polynomial. If it is the case that

$$\int_a^b f(x) \cdot p(x)\,dx = 0$$

for every polynomial p, then in particular

$$\int_a^b f(x) \cdot f(x)\,dx = 0\,.$$

Since the integrand here is continuous and nonnegative, we must conclude that $f \equiv 0$.

Now let f be an arbitrary continuous function that satisfies

$$\int_a^b f(x) \cdot p(x)\,dx = 0$$

for every polynomial p. Let $\epsilon > 0$. By the Weierstrass Approximation Theorem, select a polynomial q so that

$$|f(x) - q(x)| < \epsilon$$

for every $x \in [a, b]$. Then

$$0 = \int_a^b f(x) \cdot p(x)\,dx = \int_a^b [f(x) - q(x)] \cdot p(x)\,dx + \int_a^b q(x) \cdot p(x)\,dx \equiv I + II$$

for all polynomials p. We conclude that

$$\left|\int_a^b q(x)p(x)\,dx\right| \le \epsilon\cdot(b-a)\cdot\max_{[a,b]}|p|\,.$$

Since p is any polynomial, we may take $p = q$. Thus

$$\left|\int_a^b q^2(x)\,dx\right| \le \epsilon\cdot(b-a)\cdot\max_{[a,b]}|q|\,. \qquad (*)$$

If $\epsilon > 0$ is sufficiently small, line $(*)$ is impossible unless $q \equiv 0$. But this implies that $f \equiv 0$.

3. Restrict attention to the interval $I = [0, 1]$. The uniform limit on I of a sequence of polynomials of degree at most 10 will itself be a polynomial of degree at most 10. That rules out most continuous functions.

5. It would be impossible to approximate $f(x) = 100x$ by such polynomials.

7. Suppose that f is continuously differentiable on the interval $I = [0, 1]$. Let $\epsilon > 0$. Apply the usual Weierstrass Approximation Theorem to f' on I. So we obtain a polynomial p such that

$$|f'(x) - p(x)| < \epsilon$$

for all $x \in I$. Now define

$$P(x) = f(0) + \int_0^x p(t)\,dt\,.$$

It follows that

$$\begin{aligned}
|f(x) - P(x)| &= |f(0) + \int_0^x f'(t)\,dt - [P(0) + \int_0^x p(t)\,dt]| \\
&\le \int_0^x |f'(t) - p(t)|\,dt \\
&< \epsilon\,.
\end{aligned}$$

So we have produce a polynomial P such that

$$|f(x) - P(x)| < \epsilon$$

and

$$|f'(x) - P'(x)| < \epsilon$$

for all $x \in I$.

* **9.** Imitate the solution of Exercise 6 of Section 8.2.

* **11.** Let f be a continuous function on the square

$$S = \{(x, y) : x \in [0, 1], y \in [0, 1]\} .$$

Let $\epsilon > 0$. Then there is a polynomial

$$p(x, y) = a_{0,0} + a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{0,2}y^2 + a_{1,1}xy + \cdots + a_{j,k}x^j y^k$$

such that

$$|f(x, y) - p(x, y)| < \epsilon$$

for all $(x, y) \in S$.

We shall not prove this result, but just make the following remark. If one can prove that a continuous function $f(x, y)$ on S can be well approximated by a product $\varphi(x) \cdot \psi(y)$ of continuous functions of one variable, then the two-variable Weierstrass Approximation Theorem will follow directly from the one-variable version.

Chapter 9

Elementary Transcendental Functions

9.1 Power Series

1. Let f, g be real analytic functions such that the composition $f \circ g$ makes sense. In order to show that $f \circ g$ is real analytic we need to show that for each $x_0 \in \operatorname{dom} g$, there exist $\delta > 0$ and C and $R > 0$ such that for all $x \in [x_0 - \delta, x_0 + \delta]$,

$$\left| \frac{(f \circ g)^{(k)}(x)}{k!} \right| \leq C \frac{1}{R^k}.$$

(See the remark at the end of Section 10.2 in the text.) This will show (see Exercise 8) that the power series of $f \circ g$ at x_0 converges $f \circ g$.

We have that

$$\frac{d^k}{dx^k}(f \circ g) = \sum \frac{k!}{i!j! \cdots h!} \frac{d^m}{dx^m} f \cdot \left(\frac{g'}{1!} \right)^i \left(\frac{g''}{2!} \right)^j \cdots \left(\frac{g^{(\ell)}}{\ell!} \right)^h,$$

where $m = i + j + \cdots + h$ and the sum is taken over all integer solutions of the equation

$$i + 2j + \cdots + \ell h = k.$$

This formula is the formula for the k^{th} derivative of a composite function. Now using the estimate

$$|f^{(k)}(x)| \leq C \cdot \frac{k!}{R^k},$$

valid for all real analytic functions with suitable constants C and R, we have

$$
\begin{aligned}
|\frac{d^k}{dx^k}(f \circ g)(x)| &\leq C^{k+1} \cdot \sum \frac{k!}{i!j! \cdots h!} \cdot \frac{m!}{R^m} \cdot \frac{1}{R^i} \frac{1}{R^{2j}} \cdots \frac{1}{R^{\ell h}} \\
&\leq C^{k+1} \cdot \sum \frac{k!}{i!j! \cdots h!} \cdot \frac{m!}{R^{2m}} \\
&= C^{k+1} \frac{k!}{R^{2k}},
\end{aligned}
$$

which implies that $f \circ g$ is real analytic.

3. The series on the right certainly converges. Simply multiply the given equation on both sides by $1 - \beta$.

5. Guess a solution of the form $y = \sum_{j=0}^{\infty} a_j x^j$. Substitute this guess into the differential equation to obtain

$$\sum_{j=1}^{\infty} j a_j x^{j-1} + \sum_{j=0}^{\infty} a_j x^j = x \, .$$

Now adjust the index of summation in the first series to arrive at

$$\sum_{j=0}^{\infty} (j+1) a_{j+1} x^j + \sum_{j=0}^{\infty} a_j x^j = x \, .$$

We can combine the two series on the lefthand side to find that

$$\sum_{j=0}^{\infty} [(j+1) a_{j+1} + a_j] x^j = x \, .$$

Comparing the left and right sides, we find that

$$
\begin{aligned}
1 \cdot a_1 + a_0 &= 0 \\
2 \cdot a_2 + a_1 &= 1 \\
(j+1) a_{j+1} + a_j &= 0 \quad \text{for } j \geq 2 \, .
\end{aligned}
$$

If we set $a_0 = C$, an arbitrary constant, then we find that

$$
\begin{aligned}
a_1 &= -C \\
a_2 &= (1 + C)/2 \\
a_{j+1} &= -a_j/(j+1) \quad \text{for } j \geq 2 \, .
\end{aligned}
$$

We write out a few more terms:

$$\begin{aligned} a_3 &= -(1+C)/3! \\ a_4 &= +(1+C)/4! \end{aligned}$$

In sum, the solution we have found is

$$\begin{aligned} &C(1-x) + (1+C)\cdot(x^2/2 - x^3/3! + x^4/4! - + \cdots) \\ &\quad = \quad -1 + x + (1+C)\cdot(1 - x + x^2/2 - x^3/3! + x^4/4! - + \cdots) \\ &\quad = \quad -1 + x + (1+C)e^{-x}\,. \end{aligned}$$

This solution is certainly real analytic.

* **7.** Suppose that the zero set Z of the analytic function has an accumulation point a in the interval. In fact let us write $z_j \in Z$ and $z_j \to a$. Consider the power series expansion of f about a. Say that it is $\sum_{j=0}^{\infty} \alpha_j (x-a)^j$. Since $f(a) = 0$, we may conclude that $\alpha_0 = 0$. But

$$f'(a) = \lim_{b\to a} \frac{f(b) - f(a)}{b - a} = \lim_{j\to\infty} \frac{f(z_j) - f(a)}{z_j - a} = \lim_{j\to\infty} \frac{0 - 0}{z_j - a} = 0\,.$$

Hence $\alpha_1 = f'(a) = 0$. In like manner, we can show iteratively that $\alpha_2 = 0$, $\alpha_3 = 0$, etc. So the function is identically 0.

9.2 More on Power Series: Convergence Issues

1. We estimate the remainder term:

$$|\mathcal{R}_{k,a}(x)| \quad = \quad \left| \int_a^x f^{(k+1)}(t) \frac{(x-t)^k}{k!}\, dt \right| \le \int_a^x C \cdot \frac{(k+1)!}{R^{k+1}} \cdot \frac{|x-t|^k}{k!}\, dt\,.$$

We assume that $x \in (a-R, a+R)$. So $|x-t|/R \equiv \gamma < 1$. Thus we can estimate the last line by

$$\int_a^x C(k+1)\gamma^k\, dt \le \frac{C}{R}(k+1)\gamma^k\,.$$

We conclude that

$$\left| f(x) - \sum_{j=0}^{k} f^{(j)}(a)\frac{(x-a)^j}{j!} \right| = |\mathcal{R}_{k,a}(x)| \le \frac{C}{R}(k+1)\gamma^k \,.$$

This of course tends to 0 as $k \to \infty$.

3. The series $\sum_j x^j$ converges on $(-1,1)$.

The series $\sum_j x^j/j$ converges on $[-1,1)$.

The series $\sum_j (-x)^j/j$ converges on $(-1,1]$.

The series $\sum_j x^j/j^2$ converges on $[-1,1]$.

5. We apply the Root Test to the series

$$\sum_{j=0}^{\infty} \frac{a_j}{j+1}(x-c)^{j+1} \qquad (*)$$

to find that

$$\lim_{j\to\infty} \left| \frac{a_j}{j+1}(x-c)^{j+1} \right|^{1/j} = \lim_{j\to\infty} |a_j(x-c)^j|^{1/j} \,.$$

This shows that the series

$$\sum_{j=0}^{\infty} \frac{a_j}{j+1}(x-c)^{j+1}$$

has the same radius of convergence as

$$\sum_{j=0}^{\infty} a_j(x-c)^j \,.$$

Of course the derivative of F equals f just by term-by-term differentiation.

7. Since all the derivatives are positive, all the coefficients of the Taylor series expansion are positive. We write

$$f(x) = \sum_{j=0}^{k} f^{(j)}(a)\frac{(x-a)^j}{j!} + \mathcal{R}_{k,a}(x)$$

hence

$$\sum_{j=0}^{k} f^{(j)}(a)\frac{(x-a)^j}{j!} = f(x) - \mathcal{R}_{k,a}(x)\,.$$

But, since all the derivatives of f are positive, the remainder term is positive. So the last line implies

$$\sum_{j=0}^{k} f^{(j)}(a)\frac{(x-a)^j}{j!} < f(x)\,.$$

So we have the partial sum of the power series expansion on the left, with all positive terms as long as $x > a$, and it is bounded above by $f(x)$, independent of k. This implies that the power series converges at x. And it converges to f. Thus the function f is real analytic.

9. By Exercise 8 of Section 9.1, the function is infintely differentiable on $\mathbb{R}$ and all the derivatives at the origin are 0. Thus the power series expansion about the origin is

$$\sum_{j=0}^{\infty} 0 \cdot x^j\,.$$

This of course converges to 0 at every x. So it does *not* converge to h.

9.3 The Exponential and Trigonometric Functions

1. We know that

$$\sin(\mathrm{Sin}^{-1}x) = x$$

so that, by the Chain Rule,

$$\cos(\mathrm{Sin}^{-1}x)\cdot(\mathrm{Sin}^{-1})'(x) = 1\,.$$

As a result,

$$\sqrt{1-x^2}\cdot(\mathrm{Sin}^{-1})'(x) = 1\,.$$

Therefore

$$(\mathrm{Sin}^{-1})'(x) = \frac{1}{\sqrt{1-x^2}}\,.$$

3. We know that

$$\sin 2x = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - + \cdots .$$

And

$$\begin{aligned} 2\sin x \cos x &= 2 \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots \right) \\ &\quad \cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots \right) . \end{aligned}$$

And now it is just a matter of matching up monomials to see that the identity is true.

7. For part **(a)**, first observe that e^x is obviously positive when $x \geq 0$. To treat the case of $x < 0$, simply note that (using power series) that $e^x \cdot e^{-x} = 1$. Then, when $x < 0$,

$$e^x = \frac{1}{e^{-x}} > 0 .$$

For part **(b)**, simply plug 0 into the power series.

For part **(c)**, just differentiate the power series term by term.

9. We have

$$\cos 4x = \cos^2 2x - \sin^2 2x = (\cos^2 x - \sin^2 x)^2 - (2\sin x \cos x)^2 .$$

11. For part (a), notice that

$$\sin(s+t) = \frac{e^{i(s+t)} - e^{-i(s+t)}}{2i} = \frac{e^{is}e^{it} - e^{-is}e^{-it}}{2i} .$$

On the other hand,

$$\begin{aligned} \sin s \cos t + \cos s \sin t &= \frac{e^{is} - e^{-is}}{2i} \cdot \frac{e^{it} + e^{-it}}{2} + \frac{e^{is} + e^{-is}}{2} \cdot \frac{e^{it} - e^{-it}}{2i} \\ &= \frac{e^{is}e^{it} + e^{is}e^{-it} - e^{-is}e^{it} - e^{-is}e^{-it}}{4i} \\ &\quad + \frac{e^{is}\epsilon^{it} - e^{is}e^{-it} + e^{-is}e^{it} - e{-}ise^{-it}}{4i} \\ &= \frac{e^{is}e^{it} - e^{-is}e^{-it}}{2i} . \end{aligned}$$

In conclusion,

$$\sin(s+t) = \sin x \cos t + \cos s \sin t\,.$$

Part **(b)** is proved similarly.

For **(c)**, we calculate

$$\begin{aligned}
\cos^2 s - \sin^2 s &= \left(\frac{e^{is}+e^{-is}}{2}\right)^2 - \left(\frac{e^{is}-e^{-is}}{2i}\right)^2 \\
&= \frac{e^{2is}+2+e^{-2is}}{4} - \frac{e^{2is}-2+e^{-2is}}{-4} \\
&= \frac{e^{2is}+e^{-2is}}{2} \\
&= \cos 2s\,.
\end{aligned}$$

Part **(d)** is proved similarly.

To prove part **(e)**, we note that

$$\sin(-s) = \frac{e^{i(-s)} - e^{i(--s)}}{2i} = -\frac{e^{is}-e^{-is}}{2i} = -\sin s\,.$$

Part **(f)** is proved similarly.

Part **(g)** is proved by direct differentiation.

9.4 Logarithms and Powers of Real Numbers

1. One could do this problem by applying Stirling's formula to $j!$. A more elementary argument is this:

Notice that

$$j! \geq \underbrace{(j/2)\cdot(j/2)\cdot\cdots\cdot(j/2)}_{j/2 \text{ times}}\cdot[j/2]\cdot([j/2]-1)\cdot\cdots\cdot 3\cdot 2\cdot 1\,.$$

Hence

$$\frac{j^{j/2}}{j!} \leq \frac{2^{j/2}}{[j/2]\cdot([j/2]-1)\cdot\cdots\cdot 3\cdot 2\cdot 1} \leq 2\cdot\frac{2}{[j/2]}\cdot\frac{2}{[j/2]-1}\cdot 1\cdot 1\cdot\cdots\cdot 1 \to 0$$

as $j \to \infty$.

3. A polynomial of degree k differentiates to 0 after $k+1$ differentiations. By contrast, all the derivatives of $\log x$ are nonvanishing.

The logarithm function grows more slowly than any nonconstant polynomial.

The logarithm function only has one zero, but it is not a degree-one polynomial.

The logarithm function does not have the entire real line as its domain.

5. It is convenient to use the notation exp to denote the exponential function. We know that

$$(\ln x)' = \frac{1}{\exp'(\ln x)} = \frac{1}{x}\,.$$

That is part **(a)**.

It follows from part (a) that the logarithm function is strictly increasing.

We know that $e^0 = 1$ hence $\ln 1 = 0$.

We know that $e^1 = e$ hence $\ln e = 1$.

The graph of $y = e^x$ is asymptotic to the negative real axis, so part **(e)** follows.

We know that

$$e^{\ln s + \ln t} = e^{\ln s} \cdot e^{\ln t} = s \cdot t\,.$$

Taking logarithms of both sides gives

$$\ln s + \ln t = \ln(s \cdot t)\,.$$

We see that

$$\ln t + \ln(1/t) = \ln(t \cdot 1/t) = \ln 1 = 0$$

hence

$$\ln(1/t) = -\ln t\,.$$

Therefore

$$\ln(s/t) = \ln(s \cdot (1/t)) = \ln s + \ln(1/t) = \ln s - \ln t\,.$$

Chapter 10

Differential Equations

10.1 Picard's Existence and Uniqueness Theorem

1. Note that $x_0 = 0$ and $y_0 = 1$. We follow Picard's technique to calculate that

$$\begin{aligned}
y_1(x) &= 1 + \int_0^x 1 + t\,dt = 1 + x + \frac{x^2}{2}\,, \\
y_2(x) &= 1 + \int_0^x 1 + t + \frac{t^2}{2} + 1\,dt = 1 + x + x^2 + \frac{x^3}{6}\,, \\
y_3(x) &= 1 + \int_0^x 1 + t + t^2 + \frac{t^3}{6} + t\,dt = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\,, \\
y_4(x) &= 1 + \int_0^x 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{24} + t\,dt = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\,.
\end{aligned}$$

A pattern emerges. It is clear that these Picard iterates are converging to

$$1+x+\frac{2x^2}{2!}+\frac{2x^3}{3!}+\frac{2x^4}{4!}+\cdots = -1-x+2\left(1 + x\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = -1-x+2e^x\,.$$

So we have found the solution $y = -1 - x + 2e^x$ to our initial value problem.

3. With

$$y = \sqrt{\frac{2}{3}\ln(1+x^2) + C}\,,$$

we find that

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}\cdot\left(\frac{2}{3}\ln(1+x^2)+C\right)^{-1/2}\cdot\frac{2}{3}\cdot\frac{1}{1+x^2}\cdot 2x \\ &= \frac{2x}{3+3x^2}\left(\frac{2}{3}\ln(1+x^2)+C\right)^{-1/2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{2x}{3y+3x^2y} &= \frac{2x}{(3+3x^2)y} \\ &= \frac{2x}{(3+3x^2)}\left(\frac{2}{3}\ln(1+x^2)+C\right)^{-1/2}.\end{aligned}$$

So the differential equation is satisfied.

For the initial value problem, we solve

$$2 = y(0) = \sqrt{\frac{2}{3}\ln 1 + C}$$

so $C = 4$. Thus the particular solution is

$$y = \sqrt{\frac{2}{3}\ln(1+x^2) + 4}\,.$$

5. On the interval $[-1, 1]$ the exponential function is bounded and has bounded derivative. So the Picard iteration technique applies and the problem has a solution.

7. If we are given an nth order, linear ordinary differential equation of the form

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdot + a_2 y'' + a_1 y' + a_0 y = g(x)\,,$$

then this is equivalent to the first order, linear system

$$\begin{aligned}
y_1' &= y_2 \\
y_2' &= y_3 \\
y_3' &= y_4 \\
&\cdots \\
y_{n-1}' &= y_n \\
y_n' &= -\frac{a_0}{a_n}y_1 - \frac{a_1}{a_n}y_2 - \frac{a_2}{a_n}y_3 - \cdots - \frac{a_{n-1}}{a_n}y_n + \frac{g(x)}{a_n}\,.
\end{aligned}$$

We can think of this system as

$$(y_1, y_2, \ldots, y_n)' = F(x, y_1, y_2, \ldots, y_n)$$

or, in abbreviated form,

$$Y' = f(x, Y)\,.$$

We impose the usual conditions on F. Then one sees that, if we accept the idea of integration of vector-valued functions, both the statement and proof of the Picard Theorem go through just as in the classical setting. So we can solve vector-valued ordinary differential equations of first order. Which in turn means that we can solve nth order, linear equations.

9. Clearly the usual conditions that F be bounded and ∇F be bounded will do the trick.

* **11.** **(a)** The equations $F(x,y) = \langle -y, x\rangle$ and $\gamma'(t) = F(\gamma(t))$ translate to

$$\langle \gamma_1'(t), \gamma_2'(t)\rangle = \langle -\gamma_2(t), \gamma_1(t)\rangle\,.$$

Thus

$$\gamma_1'(t) = -\gamma_2(t)$$

and

$$\gamma_2'(t) = \gamma_1(t)\,.$$

Differentiating the first of these equations gives

$$\gamma_1''(t) = -\gamma_2'(t) = -\gamma_1(t)$$

or

$$\gamma_1''(t) + \gamma_1(t) = 0 .$$

This is a familiar equation whose solution set is

$$\gamma_1(t) = A\cos t + B\sin t .$$

Alternatively, we could differentiate the second equation to obtain

$$\gamma_2''(t) = \gamma_1'(t) = -\gamma_2(t) .$$

This becomes

$$\gamma_2''(t) + \gamma_2(t) = 0 .$$

It is convenient for us to write the general solution of this equation as

$$\gamma_2(t) = -B\cos t + A\sin t .$$

In summary, the curves

$$\gamma(t) = (A\cos t + B\sin t, -B\cos t + A\sin t)$$

satisfy

$$\gamma'(t) = F(\gamma(t)) .$$

So these are the integral curves.

(b) The equation

$$\gamma'(t) = F(\gamma(t))$$

leads to

$$(\gamma_1'(t), \gamma_2'(t)) = (\gamma_1(t) + 1, \gamma_2(t) - 2) .$$

Therefore

$$\gamma_1'(t) = \gamma_1(t) + 1$$

and

$$\gamma_2'(t) = \gamma_2(t) - 2 .$$

These two equations are easily solved to yield

$$\gamma_1(t) = -1 + Ce^t$$

and

$$\gamma_2(t) = 2 + Ce^t .$$

Thus the integral curves are

$$\gamma(t) = (-1 + Ce^t, 2 + Ce^t) .$$

10.2 Power Series Methods

1. The main point is that the coefficient $|x|$ is not real analytic so we cannot expect a real analytic solution.

3. Guess a solution of the form $y = \sum_{j=0}^{\infty} a_j x^j$. Plugging this into the differential equation yields

$$\sum_{j=1}^{\infty} j a_j x^{j-1} - \sum_{j=0}^{\infty} a_j x^{j+1} = x\,.$$

Adjusting the index of summation in both sums gives

$$\sum_{j=0}^{\infty} (j+1) a_{j+1} x^j - \sum_{j=1}^{\infty} a_{j-1} x^j = x\,.$$

With a little manipulation this becomes

$$\sum_{j=1}^{\infty} \Big[(j+1) a_{j+1} - a_{j-1} \Big] x^j = x - a_1\,.$$

Now let $a_0 = C$. From the above we read off that

$$\begin{aligned} a_0 &= C\,, \\ a_1 &= 0\,, \\ 2a_2 - a_0 = 1 \ \text{ so that } \ a_2 &= \frac{C+1}{2}\,, \\ 3a_3 - a_1 = 0 \ \text{ so that } \ a_3 &= 0\,, \\ 4a_4 - a_2 = 0 \ \text{ so that } \ a_4 &= \frac{C+1}{4 \cdot 2}\,, \\ 5a_5 - a_3 = 0 \ \text{ so that } \ a_5 &= 0\,, \\ 6a_6 - a_4 = 0 \ \text{ so that } \ a_6 &= \frac{C+1}{6 \cdot 4 \cdot 2}\,. \end{aligned}$$

The initial condition tells us that $C = 2$. So we conclude that

$$\begin{aligned} y &= 2 + \frac{3}{1!} \cdot \frac{x^2}{2} + \frac{3}{2!} \cdot \left(\frac{x^2}{2} \right)^2 + \frac{3}{3!} \cdot \left(\frac{x^2}{2} \right)^3 + \cdots \\ &= -1 + 3e^{x^2/2}\,. \end{aligned}$$

5. The Method of Power Series: We guess a solution of the form $y = \sum_{j=0}^{\infty} a_j x^j$. Plugging this guess into the differential equation yields

$$\sum_{j=1}^{\infty} j a_j x^{j-1} - \sum_{j=0}^{\infty} a_j x^j = x \,.$$

Adjusting the index in the first series yields

$$\sum_{j=0}^{\infty} (j+1) a_{j+1} x^j - \sum_{j=0}^{\infty} a_j x^j = x \,.$$

A little manipulation now yields

$$\sum_{j=0}^{\infty} \left[(j+1) a_{j+1} - a_j \right] x^j = x \,.$$

Now we may calculate the coefficients:

$$a_0 = C \,,$$

$$\begin{aligned} a_1 - a_0 = 0 \ \text{ so that } \ a_1 = a_0 = C \,, \\ 2a_2 - a_1 = 1 \ \text{ so that } \ a_2 = \frac{a_1 + 1}{2} = \frac{C+1}{2!} \,, \\ 3a_3 - a_2 = 0 \ \text{ so that } \ a_3 = \frac{a_2}{3} = \frac{C+1}{3!} \,, \\ 4a_4 - a_3 = 0 \ \text{ so that } \ a_4 = \frac{a_3}{4} = \frac{C+1}{4!} \,. \end{aligned}$$

The pattern is now clear. We see that

$$\begin{aligned} y &= C + Cx + \frac{C+1}{2!} x^2 + \frac{C+1}{3!} x^3 + \cdots \\ &= (-1 - x) + (C+1) \cdot \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) \\ &= -1 - x + (C+1) e^x \,. \end{aligned}$$

Taking the initial condition into account, we find that our solution is

$$y = -1 - x + 2e^x \,.$$

Picard's Method: We write

$$\begin{aligned} y_1 &= 1 + \int_0^x 1 + t\,dt \\ &= 1 + x + \frac{x^2}{2} \end{aligned}$$

$$\begin{aligned} y_2 &= 1 + \int_0^x 1 + t + \frac{t^2}{2} + t\,dt \\ &= 1 + x + x^2 + \frac{x^3}{6} \end{aligned}$$

$$\begin{aligned} y_3 &= 1 + \int_0^x 1 + t + t^2 + \frac{t^3}{6} + t\,dt \\ &= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} \end{aligned}$$

$$\begin{aligned} y_4 &= 1 + \int_0^x 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{24} + t\,dt \\ &= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120} \end{aligned}$$

The pattern is now clear, and we see that the solution generated by Picard is

$$y = -1 - x + 2\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = -1 - x + 2e^x\,.$$

We see that the solutions generated by the two methods coincide.

7. We guess a solution of the form $y = \sum_{j=0}^{\infty} a_j x^j$. Substituting this into the differential equation yields

$$\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} - \sum_{j=0}^{\infty} a_j x^j = x^2\,.$$

Adjusting the index of summation in the first sum gives

$$\sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}x^j - \sum_{j=0}^{\infty} a_j x^j = x^2$$

or

$$\sum_{j=0}^{\infty}\left[(j+2)(j+1)a_{j+2} - a_j\right]x^j = x^2 .$$

Now we may calculate the coefficients of the power series. We see that

$$a_0 = C ,$$

$$a_1 = D ,$$

$$2a - 2 - a_0 = 0 \ \text{ so that } \ a_2 = \frac{a_0}{2} = \frac{C}{2} ,$$

$$6a_3 - a_1 = 0 \ \text{ so that } \ a_3 = \frac{a_1}{6} = \frac{D}{6} ,$$

$$12a_4 - a_2 = 1 \ \text{ so that } \ a_4 = \frac{1+a_2}{12} = \frac{C+2}{24} ,$$

$$20a_5 - a_3 = 0 \ \text{ so that } \ a_5 = \frac{a_3}{20} = \frac{D}{120} ,$$

$$30a_6 - a_4 = 0 \ \text{ so that } \ a_6 = \frac{a_4}{30} = \frac{C+2}{720} .$$

We see that, if $D = 0$, then we get a power series solution with only even exponents.

9. We could use power series methods, but we have also learned that separation of variables is a useful technique. We rewrite the equation as

$$\frac{1}{y}\frac{dy}{dx}dx = xdx .$$

Integrating both sides gives

$$\ln y = \frac{x^2}{2} + C .$$

Exponentiating yields

$$y = e^C \cdot e^{x^2/2}$$

as the general solution.

11. We guess a solution of the form $y = \sum_{j=0}^{\infty} a_j x^j$. Substituting this guess into the differential equation yields

$$\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} + \sum_{j=0}^{\infty} 4a_j x^j = 0\,.$$

Adjusting the index of summation in the first sum gives

$$\sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}x^j + \sum_{j=0}^{\infty} 4a_j x^j = \ .$$

Some manipulation now gives

$$\sum_{j=0}^{\infty}\Big[(j+2)(j+1)a_{j+2} + 4a_j\Big]x^j = 0\,.$$

This yields the simple relation

$$(j+2)(j+1)a_{j+2} + 4a_j = 0$$

for all j.

Setting $a_0 = C$ and $a_1 = D$ we now calculate that

$$2a_2 + 4a_0 = 0 \ \text{ so that } \ a_2 = -\frac{4a_0}{2} = \frac{-4C}{2}\,,$$

$$6a_3 + 4a_1 = 0 \ \text{ so that } \ a_3 = -\frac{2}{3}a_1 = -\frac{2D}{3}\,,$$

$$12a_4 + 4a_2 = 0 \ \text{ so that } \ a_4 = \frac{-4a_2}{12} = \frac{4C}{6}\,,$$

$$20a_5 + 4a_3 = 0 \ \text{ so that } \ a_5 = \frac{-a_3}{5} = \frac{2D}{15}\,.$$

We find then that

$$\begin{aligned} y &= C\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - + \cdots +\right) \\ &\quad + \frac{D}{2}\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - + \cdots\right) \\ &= C\cos 2x + (D/2)\sin 2x\,. \end{aligned}$$

Chapter 11

Introduction to Harmonic Analysis

11.1 The Idea of Harmonic Analysis

1. We know that

$$\cos^2 x = \frac{1 + \cos 2x}{2} \,.$$

Therefore

$$\begin{aligned}
\cos^4 \theta &= (\cos^2 \theta)^2 \\
&= \left(\frac{1 + \cos 2\theta}{2}\right)^2 \\
&= \frac{1 + 2\cos 2\theta + \cos^2 2\theta}{4} \\
&= \frac{1 + 2\cos 2\theta + (1 + \cos 4\theta)/2}{4} \\
&= \frac{1}{4} + \frac{\cos 2\theta}{2} + \frac{1}{8} + \frac{\cos 4\theta}{8} \\
&= \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta \\
&= \frac{3}{8} + \frac{1}{4}(e^{i\theta} + e^{-i\theta}) + \frac{1}{16}(e^{4i\theta} + e^{-4i\theta}) \,.
\end{aligned}$$

This is the Fourier expansion for $\cos^4 \theta$.

11.2 The Elements of Fourier Series

1.

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{0}^{\frac{\pi}{2}} 1\,dx = \frac{1}{2\pi}\cdot\frac{\pi}{2} = \frac{1}{4}$$

and for $n \neq 0$, then

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{0}^{\frac{\pi}{2}} e^{-inx}dx \\
&= \frac{1}{2\pi i n}\left[1 - e^{-\frac{i\pi n}{2}}\right]
\end{aligned}$$

For $n > 0$, $n \in \mathbb{Z}$, we have that

$$\begin{aligned}
\hat{f}(n)e^{inx} + \hat{f}(-n)e^{-inx} &= \frac{1}{2\pi i n}\left[1 - e^{-\frac{i\pi n}{2}}\right]e^{inx} - \frac{1}{2\pi i n}\left[1 - e^{\frac{i\pi n}{2}}\right]e^{-inx} \\
&= \frac{1}{2\pi i n}\left[e^{inx} - e^{-inx} + e^{in\left(\frac{\pi}{2}-x\right)} - e^{in\left(x-\frac{\pi}{2}\right)}\right] \\
&= \frac{1}{\pi n}\left[\sin(nx) - \sin\left(n\left(x - \frac{\pi}{2}\right)\right)\right] \\
&= \frac{1}{\pi n}\left[\sin(nx) - \sin(nx)\cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{2}n\right)\cos(nx)\right] \\
&= \frac{1}{\pi n}\left[\left(1 - \cos\left(\frac{\pi}{2}n\right)\right)\sin(nx) + \sin\left(\frac{\pi}{2}n\right)\cos(nx)\right]
\end{aligned}$$

So the Fourier Series for $f(x)$ is:

$$\begin{aligned}
&\frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1 - \cos\left(\frac{\pi}{2}n\right)\right)\sin(nx) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{\pi}{2}n\right)\cos(nx) \\
= \; &\frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1 - \cos\left(\frac{\pi}{2}n\right)\right)\sin(nx) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}(-1)^{n+1}\cos(nx)
\end{aligned}$$

3. **(a)** $f(x) = \pi$ is its own Fourier Series.

(b)

$$f(x) = \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

(c)

$$f(x) = \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

(d)

$$\begin{aligned} f(x) &= \pi + \sin x + \cos x \\ &= \pi + \frac{e^{ix} - e^{-ix}}{2i} + \frac{e^{ix} + e^{-ix}}{2} \\ &= \pi + \left(\frac{1}{2} + \frac{1}{2i}\right) e^{ix} + \left(\frac{1}{2} - \frac{1}{2i}\right) e^{-ix} \\ &= \frac{1}{2}(1+i)e^{-ix} + \pi + \frac{1}{2}(1-i)e^{ix} \end{aligned}$$

5. The Fourier series of an even real-valued function, say $f(x)$, involves only cosines. So the Fourier series for such a function has the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx).$$

Also, for $m, n \in \mathbb{Z}$,

$$\frac{1}{2\pi} \int_0^{2\pi} \cos(mx)\cos(nx)dx = \delta_{mn}$$

where

$$\delta_{mn} = \begin{cases} 1 & , \quad m = n \\ 0 & , \quad m \neq n \end{cases}$$

is the Kronecker delta. Therefore, since

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} f(x)\cos(nx)dx$$

the identities are the Fourier series for the functions $\sin^2 x$ and $\cos^2 x$.

7. Set
$$D_N(t) = \sum_{n=-N}^{N} e^{int} .$$
Then
$$(e^{it} - 1) \cdot D_N(t)$$
telescopes so that
$$(e^{it} - 1) \cdot D_N(t) = e^{i(N+1)t} - e^{-iNt} .$$
Multiplying both sides by $e^{-it/2}$ yields
$$(e^{it/2} - e^{-it/2}) \cdot D_N(t) = e^{i(N+1/2)t} - e^{-i(N+1/2)t} .$$
Now divide out the factor on the left to obtain that
$$D_N(t) = \frac{(e^{i(N+1/2)t} - e^{-i(N+1/2)t})/(2i)}{(e^{it/2} - e^{-it/2})/(2i)} = \frac{\sin(N+1/2)t}{\sin(1/2)t} .$$

9. If we let K_N denote the kernel of σ_N, then we find that
$$\begin{aligned} K_N(x) &= \frac{1}{N+1} \sum_{j=0}^{N} D_j f(x) \\ &= \frac{1}{N+1} \sum_{j=0}^{N} \frac{\sin\left[j + \frac{1}{2}\right] x}{\sin \frac{x}{2}} \\ &= \frac{1}{N+1} \sum_{j=0}^{N} \frac{\cos jx - \cos(j+1)x}{2\sin^2 \frac{x}{2}} \end{aligned}$$
(since $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$). Of course the sum collapses and we find that
$$\begin{aligned} K_N(x) &= \frac{1}{N+1} \frac{1 - \cos(N+1)x}{2\sin^2 \frac{x}{2}} \\ &= \frac{1}{N+1} \frac{1 - [\cos^2(\frac{(N+1)x}{2}) - \sin^2(\frac{(N+1)x}{2})]}{2\sin^2 \frac{x}{2}} \\ &= \frac{1}{N+1} \frac{2\sin^2(\frac{(N+1)x}{2})}{2\sin^2 \frac{x}{2}} \\ &= \frac{1}{N+1} \left(\frac{\sin(\frac{(N+1)x}{2})}{\sin \frac{x}{2}} \right)^2 . \end{aligned}$$

11. We see that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}|K_N(t)|\,dt &= \frac{1}{2\pi}\int_0^{2\pi}K_N(t)\,dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{N+1}\sum_{j=0}^{N}D_j(t)\,dt\\ &= \frac{1}{N+1}\sum_{j=0}^{N}\frac{1}{2\pi}\int_0^{2\pi}D_j(t)\,dt\\ &= \frac{1}{N+1}(N+1)=1\,.\end{aligned}$$

11.3 An Introduction to the Fourier Transform

1. We have

$$(-x)^5\sin(-x)=x^5\sin x \quad \text{so this function is even;}$$

$$(-x)^2\sin 2(-x)=-x^2\sin 2x \quad \text{so this function is odd;}$$

$$e^{-x}\neq e^x \quad \text{and} \quad e^{-x}\neq -e^x \quad \text{so this function is neither even nor odd;}$$

$$\sin((-x)^3)=-\sin x^3 \quad \text{so this function is odd;}$$

$$\sin(-x)^2=\sin x^2 \quad \text{so this function is even;}$$

$$\cos(-x+(-x)^2)=\cos(-x+x^2)\neq\cos(x+x^2)$$

$$\text{and} \quad \cos(-x+(-x)^2)=\cos(-x+x^2)\neq-\cos(x+x^2)$$

so this function is neither even nor odd;

$$(-x)+(-x)^2+(-x)^3\neq x+x^2+x^3 \ \text{ and } \ (-x)+(-x)^2+(-x)^3\neq-(x+x^2+x^3)$$

so this function is neither even nor odd;

$$\ln\frac{1-x}{1+x}=-\ln\frac{1+x}{1-x} \quad \text{so this function is odd}\,.$$

3. We calculate that

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{-\infty}^{\infty} f(t)e^{it\xi}\,dt \\
&= \int_0^1 te^{it\xi}\,dt \\
&= \frac{t}{i\xi}e^{it\xi}\bigg|_0^1 - \int_0^1 \frac{1}{i\xi}e^{it\xi}\,dt \\
&= \frac{1}{i\xi}e^{i\xi} - \left(\frac{1}{(i\xi)^2}e^{it\xi}\right)\bigg|_0^1 \\
&= \frac{1}{i\xi}e^{i\xi} + \frac{1}{\xi^2}(e^{i\xi}-1) \\
&= e^{i\xi}\left(-\frac{i}{\xi}+\frac{1}{\xi^2}\right) - \frac{1}{\xi^2}\,.
\end{aligned}$$

5. Now

$$\hat{h}(\xi) = \int_{\mathbb{R}} (f*g)(x)e^{-2\pi i\xi x}dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)dye^{-2\pi i\xi x}dx.$$

But by Tonelli's Theorem and a change of variable,

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y)|\,|g(y)|\,dydx &= \int_{\mathbb{R}} |g(y)| \int_{\mathbb{R}} |f(x-y)|\,dxdy \\
&= \left(\int_{\mathbb{R}} |g(y)|\,dy\right)\left(\int_{\mathbb{R}} |f(z)|\,dz\right) < \infty
\end{aligned}$$

since f and g are integrable. Therefore, we can interchange the order of integration by Fubini's Theorem, to obtain:

$$\begin{aligned}
\hat{h}(\xi) &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y)g(y)dye^{-2\pi i\xi x}dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y)e^{-2\pi i\xi(x-y)}g(y)e^{-2\pi i\xi y}dxdy \\
&= \int_{\mathbb{R}} g(y)e^{-2\pi i\xi y} \int_{\mathbb{R}} f(x-y)e^{-2\pi i\xi(x-y)}dxdy \\
&= \int_{\mathbb{R}} g(y)e^{-2\pi i\xi y} \int_{\mathbb{R}} f(z)e^{-2\pi i\xi z}dzdy \\
&= \int_{\mathbb{R}} g(y)e^{-2\pi i\xi y}\hat{f}(\xi)dy \\
&= \hat{f}(\xi)\hat{g}(\xi)
\end{aligned}$$

9. Certainly we see that $e^{-x^2/2}$ is an eigenfunction of the Fourier transform with eigenvalue $\sqrt{2\pi}$. If we let $\mathcal{F}$ denote the Fourier transform, then $\mathcal{F}^4 = 4\pi^2 \cdot \mathcal{I}$, where $\mathcal{I}$ is the identity operator. So in fact there are four eigenvalues and four eigenfunctions.

11.4 Fourier Methods and Differential Equations

1. (a) As in the text, only the case $\lambda > 0$ is of interest. Since $y(0) = 0$, we conclude that

$$y(x) = A\sin\sqrt{\lambda}x\,.$$

Now, because $y(\pi/2) = 0$, we see that $\sin\sqrt{\lambda}\pi/2 = 0$. As a result,

$$\sqrt{\lambda}\pi/2 = n\pi$$

for some positive integer n so that

$$\lambda = 4n^2\,.$$

The eigenfunctions are then

$$y_n(x) = A\sin 2nx\,.$$

(c) As in the text, only the case $\lambda > 0$ is of interest. Since $y(0) = 0$, we conclude that

$$y(x) = A \sin \sqrt{\lambda} x \, .$$

Now, because $y(1) = 0$, we see that $\sin \sqrt{\lambda} \cdot 1 = 0$. As a result,

$$\sqrt{\lambda} \cdot 1 = n\pi$$

for some positive integer n so that

$$\lambda = n^2 \pi^2 \, .$$

The eigenfunctions are then

$$y_n(x) = A \sin n\pi x \, .$$

(d) As in the text, only the case $\lambda > 0$ is of interest. Since $y(0) = 0$, we conclude that

$$y(x) = A \sin \sqrt{\lambda} x \, .$$

Now, because $y(L) = 0$, we see that $\sin \sqrt{\lambda} \cdot L = 0$. As a result,

$$\sqrt{\lambda} \cdot L = n\pi$$

for some positive integer n so that

$$\lambda = n^2 \pi^2 / L^2 \, .$$

The eigenfunctions are then

$$y_n(x) = A \sin(n\pi/L) x \, .$$

3. (b) Setting $t = 0$, we see that the b_j are the coefficients for the sine series of $f(x) = (1/\pi)x(\pi - x)$. Thus

$$b_j = \frac{2}{\pi} \int_0^\pi \frac{1}{\pi} x(\pi - x) \sin jx \, dx = \frac{2}{\pi} \int_\pi x \sin jx \, dx - \frac{2}{\pi^2} \int_0^\pi x^2 \sin jx \, dx$$

$$= \frac{4}{\pi^2 j^3} [-1 + (-1)^j] \, .$$

11.5 The Heat Equation

1. The Fourier series solution to $a^2 w_{xx}(x,t) = w_t(x,t)$ satisfying the boundary conditions $w(0,t) = w(\pi,t) = 0$ is $W(x,t) = \sum_{j=1}^{\infty} b_j e^{-j^2 a^2 t} \sin jx$. This can be seen by substituting $w(x,t) = u(x)v(t)$ and separating variables. An easier way is to make the change of variables $\tau = a^2 t$ in the heat equation to obtain $w_{xx}(x,\tau) = w_\tau(x,\tau)$ having the solution $W(x,\tau) = \sum_{j=1}^{\infty} b_j e^{-j^2\tau} \sin jx$ obtained in the text, and then express the solution in terms of t.

Now let $w(x,t) = W(x,t) + g(x)$, where $g(x) = w_1 + \frac{1}{\pi}(w_2 - w_1)x$. By the superposition principal, w is also a solution to the heat equation and, since $W(0,t) = W(\pi,t) = 0$, $w(x,t)$ satisfies the boundary conditions $w(0,t) = g(0) = w_1, w(\pi,t) = g(\pi) = w_2$.

The initial temperature distribution, $w(x,0) = f(x)$, determines the values of the coefficients b_j as follows. Since $f(x) = \sum_{j=1}^{\infty} b_j \sin jx + g(x)$ the coefficients must be chosen so that $\sum_{j=1}^{\infty} b_j \sin jx = f(x) - g(x)$. Consequently, $b_j = \frac{2}{\pi}\int_0^\pi (f(x) - g(x)) \sin jx dx$, and the solution is $w(x,t) = \sum_{j=1}^{\infty} b_j e^{-j^2 a^2 t} \sin jx + g(x)$.

3. We seek separated solutions to the heat equation: $a^2 w_{xx} = w_t$, satisfying the boundary conditions $w_x(0,t) = 0 = w_x(\pi,t)$. Substitute $w(x,t) = \alpha(x)\beta(t)$ to get $a^2\alpha''\beta = \alpha\beta'$ or $\frac{\alpha''}{\alpha} = \frac{\beta'}{\alpha^2\beta}$. Thus there is a constant K such that $\frac{\alpha''}{\alpha} = K = \frac{\beta'}{a^2\beta}$. That is, $\alpha'' = K\alpha$ and $\beta' = Ka^2\beta$, so $\beta(t) = Ce^{Ka^2 t}$. Since the temperature is not expected to grow exponentially with time we assume $K \le 0$ so $\alpha(x) = \sin\sqrt{-K}x$ or $\alpha(x) = \cos\sqrt{-K}x$ or $\alpha(x) = C$, a constant. The last possibility corresponds to $K = 0$.

The boundary conditions require $\alpha'(0) = 0 = \alpha'(\pi)$. Consequently $\alpha(x) = C$, a constant, or $\alpha(x) = \cos\sqrt{-K}x$ with K chosen so that $\alpha'(0) = -\sqrt{K}\sin\sqrt{-K}\pi = 0$. Therefore, the eigenvalues are $K = -n^2, n = 0, 1, 2, \cdots$. The separated solutions are $w(x,t) = e^{-n^2 a^2 t}\cos nx$. Therefore, the series solution is $w(x,t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j e^{-j^2 a^2 t}\cos jx$ where the coefficients a_j satisfy $w(x,0) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos jx = f(x)$. That is, $a_j = \frac{2}{\pi}\int_0^\pi f(x)\cos jx dx$.

9. Let the circle C be centered at (x_0, y_0) (Cartesian coordinates) with radius R. The function $u(x,y) = w(x_0 + x, y_0 + y)$ is harmonic on the disk centered at the origin of radius R. According to the Poisson integral formula for this disk (Exercise **8**), u's value at the center of the disk: $(0, \theta)$, (polar coordinates) is given by $u(0, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(R, \phi) d\phi$. In terms of the original function w this formula can be expressed in the following form.

$$w(x_0, y_0) = \frac{1}{2\pi R} \int_{-pi} \pi w(x_0 + R\cos\phi, y_0 + R\sin\phi) R d\phi.$$

Chapter 12

Functions of Several Variables

12.1 A New Look at the Basic Concepts of Analysis

1. Let $\mathbf{s}, \mathbf{t}, \mathbf{u} \in \mathbb{R}^k$. Assume that these three points are colinear. By rotating and translating the line in space, we may as well suppose that the three points lie on the positive real axis.

Then the classical Triangle Inequality tells us that

$$|\mathbf{s} - \mathbf{t}| = |(\mathbf{s} - \mathbf{u}) + (\mathbf{u} - \mathbf{t})| \leq |\mathbf{s} - \mathbf{u}| + |\mathbf{u} - \mathbf{t}| = \|\mathbf{s} - \mathbf{u}\| + |\mathbf{u} - \mathbf{t}\| .$$

Now suppose that $\mathbf{s}, \mathbf{t}, \mathbf{u}$ are in general position—not colinear. Imagine that the points are as shown in Figure 12.1.

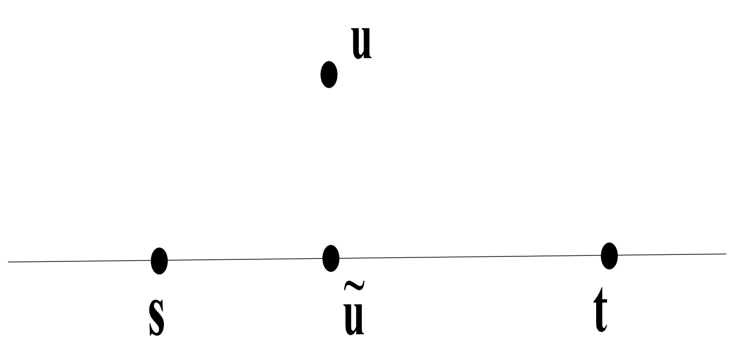

Figure 12.1: The triangle inequality.

Project the point $\mathbf{u}$ to a point $\widetilde{\mathbf{u}}$ on the line through $\mathbf{s}$ and $\mathbf{t}$. Then we know, by the result of the first paragraph, that

$$\|\mathbf{s}-\mathbf{t}\| \leq \|\mathbf{s}-\widetilde{\mathbf{u}}\| + \|\widetilde{\mathbf{u}}-\mathbf{t}\| \,.$$

But this in turn is

$$\leq \|\mathbf{s}-\mathbf{u}\| + \|\mathbf{u}-\mathbf{t}\| \,.$$

3. Just as an instance, let us prove that

$$\lim_{\mathbf{x}\to\mathbf{P}} [f(\mathbf{x}) + g(\mathbf{x})] = \lim_{\mathbf{x}\to\mathbf{P}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{P}} g(\mathbf{x}) \,.$$

Let $\epsilon > 0$. Choose $\delta_1 > 0$ so that, when $\|\mathbf{x}-\mathbf{P}\| < \delta_1$, then $|f(\mathbf{x}) - f(\mathbf{P})| < \epsilon/2$. Likewise, choose $\delta_2 > 0$ so that, when $\|\mathbf{x}-\mathbf{P}\| < \delta_2$, then $|g(\mathbf{x}) - g(\mathbf{P})| < \epsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $\|\mathbf{x}-\mathbf{P}\| < \delta$, we have

$$|[f(\mathbf{x})+g(\mathbf{x})]-[f(\mathbf{P})+g(\mathbf{P})]| \leq |f(\mathbf{x})-f(\mathbf{P})|+|g(\mathbf{x})-g(\mathbf{P})| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon \,.$$

That establishes the result.

5. We say that $f_j(\mathbf{x}) \to f(\mathbf{x})$ uniformly if, given $\epsilon > 0$, there is a J so large that, when $j \geq J$, then

$$|f_j(\mathbf{x}) - f(\mathbf{x})| < \epsilon$$

for all x. Now let us prove that, if the f_j are continuous on $\mathbb{R}^k$, then so is f.

Let $\mathbf{P} \in \mathbb{R}^k$ and $\epsilon > 0$. Choose J so large that, if $j \geq J$, then $|f_j(\mathbf{x}) - f(\mathbf{x})| < \epsilon/3$ for all $\mathbf{x}$. Choose $\delta > 0$ so that, if $|\mathbf{x}-\mathbf{P}| < \delta$, then $|f_J(\mathbf{x}) - f_J(\mathbf{P})| < \epsilon/3$. Then, for such x,

$$\begin{aligned} |f(\mathbf{x}) - f(\mathbf{P})| &\leq |f(\mathbf{x}) - f_J(\mathbf{x})| + |f_J(\mathbf{x}) - f_J(\mathbf{P})| + |f_J(\mathbf{P}) - f(\mathbf{P})| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \,. \end{aligned}$$

So f is continuous at P.

7. We treat the maximum. The minimum is handled similarly.

Let E be compact and let f be a continuous function on E. Let $M = \sup\{f(x) : x \in E\}$. Then there is a sequence $x_j \in E$ such that $f(x_j) \to M$. By compactness, there is a subsequence $\{x_{j_k}\}$ so that $x_{j_k} \to x_0 \in E$. But then, by continuity, $f(x_{j_k}) \to f(x_0) = M$. So f assumes its maximum value at $x_0 \in E$.

9. For j, k integers, let

$$E_{j,k} = \left\{ (x,y) \in \mathbb{R}^2 : (x-j)^2 + (y-k)^2 \leq \frac{1}{4} \right\} .$$

Define

$$S = \bigcup_{(j,k)\in\mathbb{Z}\times\mathbb{Z}} E_{j,k} .$$

Then each $E_{j,k}$ is a connected component of S. There are infinitely many of them.

11. We see that

$$\|\mathbf{s}\| = \|(-\mathbf{t}) + (\mathbf{s}+\mathbf{t})\| \leq \|-\mathbf{t}\| + \|\mathbf{s}+\mathbf{t}\| = \|\mathbf{t}\| + \|\mathbf{s}+\mathbf{t}\| .$$

Therefore

$$\|\mathbf{s}\| - \|\mathbf{t}\| \leq \|\mathbf{s}+\mathbf{t}\| .$$

12.2 Properties of the Derivative

1. The sum rule says that

$$\Big[f(\mathbf{x}) + g(\mathbf{x}) \Big]' = f'(\mathbf{x}) + g'(\mathbf{x}) .$$

The product rule says that

$$\Big[f(\mathbf{x}) \cdot g(\mathbf{x}) \Big]' = f'(\mathbf{x}) \cdot g(\mathbf{x}) + f(\mathbf{x}) \cdot g'(\mathbf{x}) .$$

The quotient rule says that

$$\left[\frac{f}{g}(\mathbf{x}) \right]' = \frac{g(\mathbf{x}) \cdot f'(\mathbf{x}) - f(\mathbf{x}) \cdot g'(\mathbf{x})}{g^2(\mathbf{x})}$$

provided that g does not vanish.

3. We say that f has a second derivative if its first derivative is differentiable, that is

$$\mathbf{P} \longmapsto M_{\mathbf{P}}(f)$$

possesses the first derivative. That is, we require

$$M_{\mathbf{P}+\mathbf{h}}(f) = M_{\mathbf{P}}(f) + L_{\mathbf{P}}((M(f))\mathbf{h} + R_{\mathbf{P}}(M(f)), \mathbf{h}).$$

Here $L_{\mathbf{P}}(M(f))$ is a linear map from $\mathbb{R}^k$ to $\mathbb{R}^k$. Notice that, if we write $\mathbf{e}_j = (0, \ldots, 1, \ldots, 0)$, then

$$\begin{aligned} M_{\mathbf{P}+k\mathbf{e}_j}(f) - M_{\mathbf{P}}(f) &= kL_{\mathbf{P}}(M(f))\mathbf{e}_j + R_{\mathbf{P}} \\ &= kL_{\mathbf{P}}(M(f))_j + R_{\mathbf{P}}, \end{aligned}$$

where $L_{\mathbf{P}}(M(f))_j$ is the j^{th} column of that matrix. Dividing both sides by k and letting $k \to 0$ we find that

$$\left(\frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_1}(\mathbf{P}), \ldots, \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_n}(\mathbf{P}), \right) = L_{\mathbf{P}}(M(f))_j.$$

Thus,

$$L_{\mathbf{P}}(M(f)) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{P}) \right)_{i,j=1}^{k},$$

the Hessian matrix of all second partial derivatives.

5. Suppose $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ is differentiable at a point $\mathbf{P}$. Then for each j

$$f_j(\mathbf{P} + \mathbf{h}) = f_j(\mathbf{P}) + M_j(f) \cdot \mathbf{h} + R_{j\mathbf{P}}(f, \mathbf{h})$$

with

$$\lim_{\mathbf{h} \longrightarrow 0} \frac{R_{j\mathbf{P}}(f, \mathbf{h})}{\|\mathbf{h}\|} = 0.$$

Thus

$$\begin{aligned} \mathbf{f}(\mathbf{P} + \mathbf{h}) &= {}^t(f_1(\mathbf{P}), \ldots, f_n(\mathbf{P})) \\ &\quad + {}^t(M_{1\mathbf{P}}(f) \cdot \mathbf{h}, \ldots, M_{n\mathbf{P}} \cdot \mathbf{h}) \\ &\quad + {}^t(R_{1\mathbf{P}}(f, \mathbf{h}), \ldots, R_{n\mathbf{P}}(f, \mathbf{h})) \\ &= \mathbf{f}(\mathbf{P}) + \mathbf{M}_{\mathbf{P}} + \mathbf{R}_{\mathbf{P}}(f, \mathbf{h}) \end{aligned}$$

where

$$\begin{aligned}\lim_{\mathbf{h}\to 0}\frac{\|\mathbf{R}_{\mathbf{P}}(f,\mathbf{h})\|}{\|\mathbf{h}\|} &\leq \sum_{j=1}^{n}\lim_{\mathbf{h}\to 0}\frac{|R_{j\mathbf{P}}(f,\mathbf{h})|}{\|\mathbf{h}\|}\\ &= 0\end{aligned}$$

and $M_{\mathbf{P}}$ is an $n \times m$ matrix. The converse is obvious.

7. If $M_{\mathbf{t}}$ is the zero matrix for every $\mathbf{t} \in B(\mathbf{P}, r)$, then every partial derivative is equal to 0. If $\mathbf{t}$ is an arbitrary point of the ball, then

$$\begin{aligned}f(\mathbf{t}) - f(\mathbf{0}) &= \int_0^{t_k}\frac{\partial f}{\partial t_k}(t_1, t_2, t_3, \ldots, t_{k-1}, s)\,ds\\ &\quad + \int_0^{t_{k-1}}\frac{\partial f}{\partial t_{k-1}}(t_1, t_2, t_3, \ldots, t_{k-2}, s, 0)\,ds\\ &\quad + \cdots + \int_0^{t_1}\frac{\partial f}{\partial t_1}(s, 0, 0, \ldots, 0)\,ds\,.\end{aligned}$$

Since the partial derivatives are all 0, each of the integrals on the right is equal to 0. Hence $f(\mathbf{t}) = f(\mathbf{0})$ for any $\mathbf{t}$ in the ball. In other words, f is constant.

9. Write

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))\,.$$

The rows of the new matrix $M_{\mathbf{P}}$ are simply the derivatives, as defined in the text, of the functions f_j, $j = 1, 2, \ldots, m$.

* **13.** It is enough to show that if $f : [0, 1] \to \mathbb{R}^2$ is continuous and is differentiable on $(0, 1)$ then it does not necessarily follow that there is a $\xi \in (0, 1)$ such that

$$f'(\xi) = \frac{f(1) - f(0)}{1 - 0}.$$

As an example, we take $f(t) = (t^2 + t, t^2 - t^3)$. Then $f(1) - f(0) - (2, 0)$ and $f'(t) = (2t + 1, 2t - 3t^2)$. Clearly there is no value of t in $(0, 1)$ such that $(2t + 1, 2t - 3t^2) = (2, 0)$.

12.3 The Inverse and Implicit Function Theorems

1. The image of an open set under a homeomorphism is open.

3. If $P = (x, \varphi(x)) \in \mathcal{U}$, then $\mathbf{v} = \langle -\varphi'(x), 1\rangle$ is a normal vector to $\mathcal{U}$ at P. Consider the mapping

$$T : (x,t) \longmapsto \langle x, \varphi(x)\rangle + t\langle -\varphi'(x), 1\rangle = \langle x - t\varphi'(x), \varphi(x) + t\rangle$$

for x in the domain of φ and t small. We see that we are mapping each point P to a point t units along the normal from P.

We calculate the Jacobian matrix determinant of T at a point (x, t):

$$\det \begin{pmatrix} 1 - t\varphi''(x) & \varphi'(x) \\ -\varphi'(x) & 1 \end{pmatrix} = (1 - t\varphi''(x)) + [\varphi'(x)]^2 .$$

Plainly, if t is small, then this Jacobian determinant is positive. So the inverse function theorem applies and we see that we can recover from each point Q near $\mathcal{U}$ a pair (x, t). That means that $P = (x, \varphi(x))$ is the nearest point in $\mathcal{U}$ to Q and t is the distance of Q to P.

5. If the logarithm function had two zeros then it would have a local maximum. And the Implicit Function Theorem does not apply in a neighborhood of such a maximum. But in fact the derivative of the logarithm function is always positive, so this situation does not occur.

7. Consider the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (x, y^3)$. This mapping is certainly invertible, but its Jacobian determinant at the origin is 0.

9. Let $p(x)$ be a polynomial of degree at least 1 in one real variable. Define

$$G(x, y) = y - p(x) .$$

At the point $(x, 0)$ we may calculate the partial derivative of G in x. For most x this will not be zero. So we can solve for x in terms of y. At $y = 0$ this gives a root of the polynomial.